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FRACTIONAL DERIVATIVES OF THE MULTIPLE HYPERGEOMETRIC FUNCTIONS OF FOUR VARIABLES

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ABSTRACT

In the present paper, we derive fractional derivatives involving hypergeometric functions of four variables of H. Exton [5,6,7] and C.Sharma and C.L. Parihar [11].

1. INTRODUCTION Motivated by earlier work of Srivastava and Goyal [18], recently Srivastava, Chandel and Vishwakarma [19] derived a number of key formulas for the fractional derivatives of the multivariable H-function of Srivastava and Panda [13-15] (for multivariable H-function also see Srivastava, Gupta and Goyal [16]). Each of these formulas can be shown to yield interesting new results for various classes of generalised hypergeometric functions of several variables of Srivastava and Daoust [12] (Also see Srivastava and Manocha [17, pp.64-65]), Lauricella [9], Exton [4,7], Chandel [1], Chandel and Gupta [2] and Karlsson [8]. For fractional derivatives of confluent hypergeometric forms of Karlsson's multiple hypergeometric function ${}^{(k)}F_{CD}^{(n)}$ [8] also see Chandel and Vishwakarma [3].

In the present paper, for special interest, we apply same techniques in order to derive fractional derivatives involving multiple hypergeometric functions of four variables $K_1, \ldots, K_{21}$ of Exton [5,6,7] and those functions of Sharma and Parihar [11], which are not included in Exton's functions [5,6,7].

2. FRACTIONAL DERIVATIVES INVOLVING ONE FRACTIONAL DERIVATIVE OPERATOR.

Making an appeal to the formula [10,p.69]

$$(2.1)\quad D_x^{\mu}\{x^{\lambda}\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\, x^{\lambda-\mu}, \qquad \mathrm{Re}(\lambda) > -1,$$

we have

$$D_t^{\lambda_4 - \mu_4} \{t^{\lambda_4 - 1} K_2 (a_1, a_1, a_1, \mu_4, b_1, b_1, b_1, b_1; c_1, c_2, c_3, c_4; x, y, z, t)\}$$

$$= \sum_{m, n, p, q = 0}^{\infty} D_t^{\lambda_4 - \mu_4} \left\{ t^{\lambda_4 - 1} \frac{(a_1)_{m+n+p} (\mu_4)_q (b_1)_{m+n+p+q}}{(c_1)_m (c_2)_n (c_3)_p) (c_4)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{t^q}{q!} \right\}.$$

Therefore

(2.2) $$D_t^{\lambda_4 - \mu_4} \{t^{\lambda_4 - 1} K_2 (a_1, a_1, a_1, \mu_4, b_1, b_1, b_1, b_1; c_1, c_2, c_3, c_4; x, y, z, t)\}$$

$$= \frac{\Gamma(\lambda_4)}{\Gamma(\mu_4)} t^{\mu_4 - 1} K_2 (a_1, a_1, a_1, \lambda_4, b_1, b_1, b_1, b_1; c_1, c_2, c_3, c_4; x, y, z\ t),$$

$\mathrm{Re}(\lambda_4) > 0$.

Applying the same techniques, we have also derived the fractional derivatives of K_{11}, K_{15} of Exton [5,6,7] and $F_5^{(4)}$, $F_{42}^{(4)}$, $F_{68}^{(4)}$ of Sharma and Parihar [11].

3. USE OF TWO FRACTIONAL DERIVATIVE OPERATORS

In this section, we derive the following relation by using two fractional derivative operators :

(3.1) $$D_y^{\lambda_2 - \mu_2} D_t^{\lambda_4 - \mu_4} \{y^{\lambda_2 - 1} t^{\lambda_4 - 1} K_4 (a_1, a_1, a_2, a_2, b_1, b_1, b_1, b_1; c_1, \lambda_2, c_1, \lambda_4; x, y, z, t)\}$$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_4)} y^{\mu_2 - 1} t^{\mu_4 - 1} K_4 (a_1, a_1, a_2, a_2, b_1, b_1, b_1, b_1; c_1, \mu_2, c_1, \mu_4; x, y, z, t),$$

$\mathrm{Re}(\lambda_2) > 0$, $\mathrm{Re}(\lambda_4) > 0$.

Similarly, we have also derived the results for K_7, K_8, K_{13}, K_{19}, K_{21} of Exton [5,6,7] and $F_4^{(4)}$, $F_8^{(4)}$, $F_{14}^{(4)}$, $F_{16}^{(4)}$, $F_{17}^{(4)}$, $F_{21}^{(4)}$, $F_{27}^{(4)}$, $F_{28}^{(4)}$, $F_{29}^{(4)}$, $F_{30}^{(4)}$, $F_{31}^{(4)}$, $F_{32}^{(4)}$, $F_{36}^{(4)}$, $F_{37}^{(4)}$, $F_{41}^{(4)}$, $F_{43}^{(4)}$, $F_{48}^{(4)}$, $F_{49}^{(4)}$, $F_{50}^{(4)}$, $F_{51}^{(4)}$, $F_{53}^{(4)}$, $F_{54}^{(4)}$, $F_{55}^{(4)}$, $F_{56}^{(4)}$, $F_{62}^{(4)}$, $F_{63}^{(4)}$, $F_{64}^{(4)}$, $F_{67}^{(4)}$, $F_{72}^{(4)}$, $F_{80}^{(4)}$ of Sharma and Parihar [11].

4. USE OF THREE FRACTIONAL DERIVATIVE OPERATORS

In this section, we derive the following relation by making an appeal to three fractional derivative operators :

(4.1) $D_y^{\lambda_2 - \mu_2} D_z^{\lambda_3 - \mu_3} D_t^{\lambda_4 - \mu_4} \{y^{\lambda_2 - 1} z^{\lambda_3 - 1} t^{\lambda_4 - 1}$

$$K_1(a_1, a_1, a_1, \mu_4, b_1, b_1, b_1, b_1; c_1, \lambda_2, \lambda_3, c_1; x, y, z, t)\}$$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_3)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_3)\,\Gamma(\mu_4)} y^{\mu_2 - 1} z^{\mu_3 - 1} t^{\mu_4 - 1}$$

$$K_1(a_1, a_1, a_1, \lambda_4, b_1, b_1, b_1, b_1; c_1, \mu_2, \mu_3, c_1; x, y, z, t),$$

$\mathrm{Re}(\lambda_2) > 0$, $\mathrm{Re}(\lambda_3) > 0$; $\mathrm{Re}(\lambda_4) > 0$.

On similar lines we have also derived the results involving K_2, K_6, K_{14} of Exton [5,6,7] and $F_6^{(4)}$, $F_{14}^{(4)}$, $F_{18}^{(4)}$, $F_{19}^{(4)}$, $F_{20}^{(4)}$, $F_{44}^{(4)}$, $F_{45}^{(4)}$, $F_{61}^{(4)}$, $F_{71}^{(4)}$ of Sharma and Parihar [11].

5. USE OF FOUR FRACTIONAL DERIVATIVE OPERATORS

In this section, making an appeal to four fractional derivative operators, we derive

(5.1) $D_x^{\lambda_1 - \mu_1} D_y^{\lambda_2 - \mu_2} D_z^{\lambda_3 - \mu_3} D_t^{\lambda_4 - \mu_4} \{x^{\lambda_1 - 1} y^{\lambda_2 - 1} z^{\lambda_3 - 1} t^{\lambda_4 - 1}$

$$= K_2\,(a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_1; \lambda_1, \lambda_2, \lambda_3, \lambda_4; x, y, z, t)\}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} x^{\mu_1 - 1} y^{\mu_2 - 1} z^{\mu_3 - 1} t^{\mu_4 - 1}$$

$$K_2(a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_1; \mu_1, \mu_2, \mu_3, \mu_4; x, y, z, t),$$

$\mathrm{Re}(\lambda_1) > 0$, $\mathrm{Re}(\lambda_2) > 0$, $\mathrm{Re}(\lambda_3) > 0$, $\mathrm{Re}(\lambda_4) > 0$.

We have also obtained similar results for K_5, K_{10}, K_{12}, K_{13}, K_{15}, K_{20}, K_{21} of Exton [5,6,7] and $F_5^{(4)}$, $F_6^{(4)}$, $F_7^{(4)}$, $F_8^{(4)}$, $F_{16}^{(4)}$, $F_{21}^{(4)}$, $F_{33}^{(4)}$, $F_{37}^{(4)}$ $F_{46}^{(4)}$, $F_{52}^{(4)}$, $F_{56}^{(4)}$, $F_{66}^{(4)}$, $F_{67}^{(4)}$, $F_{75}^{(4)}$, $F_{76}^{(4)}$, $F_{78}^{(4)}$ of Sharma and Parihar [11] but due to lack of space we have not presented them here.

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