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SOME RESULTS ON EIGENVALUE PROBLEMS AND BEST APPROXIMATIONS

By

A. CARBONE

Universitá Degli Studi Della Calabria

Dipartimento Di Matematica

87036 Arcavacata Di Rende (Cosenza), Italy

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ABSTRACT

In this paper, using the Ky Fan best approximation theorem, a theorem on eigenvalue problem is proved. A few simple fixed point theorems are derived as corollaries. In the end, a random analogue of nonlinear eigenvalue problem is also given.

There are interesting results dealing with the eigenvalue problems. The well-known result on best approximation is due to Ky Fan and it has many applications in fixed point theory and approximation theory.

In this paper a solution of the nonlinear eigenvalue problem $f(x) = \lambda gx$ is discussed. A few fixed point theorems are also derived. In the end, random fixed point theorems are used to deal with a random analogue of the nonlinear problem. We obtain a general and unifying result on best approximation and then derive a few fixed point theorems as corollaries.

We need the following preliminaries :

Let B_r denote a closed ball of radius r and centre 0 in a Banach space X and ∂B_r the boundary of B_r.

The following is a well-known result due to Ky Fan, known as the best approximation theorem [4].

Theorem F : *Let C be a nonempty compact convex subset of a Banach space X and $f : C \to X$ a continuous function. Then there is a $y \in C$ such that*

$$\|y - fy\| = d(fy, C) = \inf\{\|fy - z\| : z \in C\}.$$

Note : In theorem F if $f : C \to C$ then the well-known Schauder fixed point theorem is obtained. For further results see [9], [8], [3] and [5].

We prove the following and then derive nonlinear eigenvalue problem and fixed point theorems.

Theorem 1. *Let B_r be a ball in a finite dimensional Banach space X and $f, g : B_r \to X$ continuous functions. If $hx = f(x) - \mu gx + Ix$, where μ is any constant and I is an identity function on X then there is $y \in B_r$ such that*

$$\|y - hy\| = d(hy, B_r).$$

Proof : Let $R : X \to B_r$ be a radial retraction, that is

$$Rx = x \ \text{ if } \ \|x\| \le r$$
$$= \frac{rx}{\|x\|} \ \text{ if } \ \|x\| \ge r.$$

Then R is a continuous function.

Let $h_1 x = R.hx$. Then $h_1 : B_r \to B_r$ is a continuous function. By the Brouwer fixed point theorem h_1 has a fixed point, say $y \in B_r$ such that $h_1 y = y$.

Now $y = h_1 y = R.hy = hy$ if $\|hy\| \le r$, so y is a fixed point of h. Also

$$y = h_1 y = R.hy = \frac{rhy}{\|hy\|}, \ \text{ if } \ \|hy\| \ge r.$$

So

$$\|y - hy\| = 0, \ \text{if } \ \|hy\| \le r$$

and

$$\|y - hy\| = \|hy - \frac{rhy}{\|hy\|}\|$$
$$= \|hy\| - r, \ \text{ if } \ \|hy\| \ge r.$$

For any $x \in B_r$ we have

$$\|y - hy\| = \|hy\| - r \le \|hy\| - \|x\|$$
$$\le \|x - hy\|,$$

so $\|y - hy\| \le \|x - hy\|$ for all $x \in B_r$, that is, $\|y - hy\| = d(hy, B_r)$.

Now we give a few particular cases.

In case $g = I$ and $\mu = 1$ then we get the following known results in fixed point theory.

1. If in Theorem 1, $g = I$ and $\mu = 1$, and $f : B_r \to X$ a continuous function such that $f(\partial B_r) \subset B_r$ then f has a fixed point.

2. If in Theorem 1, $g = I$ and $\mu = 1$, and $f : B_r \to X$ is a continuous function satisfying the following :

In case $y \neq fy$ then the line segment $[y, fy]$ has at least two points of B_r,

then f has a fixed point.

It is easy to see that f has a fixed point. Let $y \neq fy$. Then by Theorem 1 there is a $y \in B_r$ such that $\|y - fy\| = d(fy, B_r)$.

Let $z \in B_r$. Then $z = \lambda y + (1 - y) fy$, $0 < \lambda < 1$.

Now,

$$\|z - fy\| = \|\lambda y + (1 - \lambda)fy - fy\|$$
$$= \|\lambda(y - fy\| = |\lambda| \; \|y - fy\| < \|y - fy\| = d(fy, B_r);$$

a contradiction.

Hence $y = fy$.

3. If in Theorem 1, $g = I, \mu = 1$ and $f : B_r \to X$ continuous function satisfying the following :

if $y \in \partial B_r$ and $y \neq fy$ then there is a λ (real or complex depending on the field selected) such that $z = \lambda y + (1 - \lambda)fy$ and $|\lambda| < 1$,

then f has a fixed point.

Indeed, as in (2) we could easily show that f has a fixed point.

The following deals with the nonlinear eigenvalue problem (see also [1]).

Theorem 2. If $f, g : B_r \to X$ are continuous functions and μ is any constant satisfying the following condition :

if for $y \in \partial B_r$ the equation $fy = \alpha y + \mu g y$ holds then $\alpha \leq 0$,

then there is a $y_0 \in B_r$ such that $fy_0 = \mu g y_0$.

Proof : Let $hy = fy - \mu gy + Iy$.

Then by Theorem 1 we have

$$\|y - hy\| = d(hy, B_r).$$

If $y \in \partial B_r$ and $y \neq hy$ then $\|hy\| > r$.

But then

$$fy - \mu gy = \alpha y$$

or

$$fy - \mu gy + Iy = (\alpha + 1)y$$

will give

$$\frac{\|fy - \mu gy + Iy\|}{\|y\|} = \alpha + 1 > 1,$$

(since $\frac{\|hy\|}{\|y\|} > 1$)), that is, $\alpha > 0$ contradiction.

Hence $y = hy$, that is, $fy = \mu gy$.

In case $g = I$ an identity function then we get the following :

Corollary 1 : If $f : B_r \to X$ is a continuous function, and μ is any constant, satisfying the following : if for $y \in \partial B_r$ the equation $fy = (\mu + \alpha)y$ holds for $\alpha \le 0$,
then $fy = \mu y$.

This follows from Theorem 2 by taking $g = I$.

Theorem 3. *If f_1 and $g : B_r \to X$ are continuous functions and μ is any constant satisfying the following :*

if for $y \in \partial B_r$ the equation $f_1 y = \mu gy + \alpha y$ holds then $\alpha \ge 0$,

then $f_1 y = \mu gy$.

Proof : As in Theorem 2 we have $fy = \mu gy$ we get the result by just putting $f_1 = 2\mu g - f$. In this way f_1 and g of Theorem 3 satisfy conditions as f and g satisfy in Theorem 2. Hence the result.

Definition : If C is a subset of a Banach space X then the inward set of C at $x \in C$, denoted by $I_C(x)$, is defined by

$$I_C(x) = \{z \in X : z = x + r(y - x) \text{ for } y \in C \text{ and } r > 0\}.$$

$\overline{I_C(x)}$ stands for the closure of $I_C(x)$. A function $f : C \to X$ is an inward map if $f(x) \in I_C(x)$ for all $x \in C$.

f is weakly inward if $fx \in \overline{I_C(x)}$ for every $x \in C$.

The outward set $O_C(x)$ is defined as follows :

$$O_C(x) = \{z \in X : z = x + r(y - x) \text{ for } y \in C \text{ and } r < 0\}$$

For details see Browder [3] and Halpern and Bergman [5].

Theorem 4. *Let B_r be a ball in a finite dimensional Banach space X and $f, g : B_r \to X$ continuous functions. Then there exists a $y \in B_r$ such that*

$$\|y - hy\| = d(hy, \overline{I_{B_r}(y)})$$

where $hy = fy - \mu gy + Iy$ and μ is any constant.

Proof : By Theorem 1 there is a $y \in B_r$ such that

$$\|y - hy\| = d(hy, B_r).$$

If $\|y - hy\| = 0$ then $y = hy$. Assume that $\|y - hy\| > 0$. We show that $\|y - hy\| \le \|hy - z\|$ for all $z \in \overline{I_{B_r}(y)}$. Let

$x \in \overline{I_{B_r}}(y) \setminus B_r$. Then there is a $u \in B_r$ and $r > 1$ such that $x = y + r(u - y)$.

Suppose $\|y - hy\| > \|hy - x\|$ and seek a contradiction. Now, $\frac{1}{r}x = \frac{1}{r}y + u - y$, that is $u = \frac{1}{r}x + (1 - \frac{1}{r})y \in B_r$ so

$$\|hy - u\| = \|hy - \frac{1}{r}x - (1 - \frac{1}{r})y\|$$

$$= (1 - \frac{1}{r})\|hy - y\| + \frac{1}{r}\|hy - x\|$$

$$< \|hy - y\|(1 - \frac{1}{r}) + \frac{1}{r}\|hy - y\|$$

$$= \|hy - y\|, \text{ a contradiction.}$$

Hence $\|y - hy\| \le \|hy - x\|$ for $x \in \overline{I_{B_r}(y)} \setminus B_r$.

Since norm is continuous so

$$\|y - hy\| \le \|hy - z\| \text{ for all } z \in \overline{I_{B_r}(y)}.$$

It is easy to derive fixed point theorems as corollaries for inward maps.

The following corollary is for the nonlinear eigenvalue problems.

Corollary 2 : If in Theorem 4, additionally f and g satisfy the following :

if for $y \in \partial B_r$ the equation

$$fy = \alpha y + \mu gy$$

holds then $\alpha \le 0$,

then there is a $y \in \overline{I_{B_r}(y)}$ such that

$$fy = \mu gy.$$

Proof : If we write $hy = fy - \mu gy + y$, then by Theorem 4

$$\|y - hy\| = d(hy, \overline{I_{B_r}(y)}).$$

If $y \in \partial B_r$ and $y \ne hy$ then $\|hy\| > r$ and $fy = \alpha y + \mu gy$ for $\alpha \le 0$ will lead to a contradiction. Hence

$$y = hy, \text{ that is, } fy = \mu gy.$$

Here we give a random analogue of the nonlinear eigenvalue problem.

Let (Ω, β) be a measurable space with β a σ-algebra of subsets of Ω.

A mapping $f : \Omega \times B_r \to X$ is continuous if for each fixed $\omega \in \Omega$, $f(w, .) : B_r \to X$ is continuous.

The following random fixed point theorem, due to Sehgal and Waters [6], extends results given in [2]. We will need this theorem in our work.

Theorem 5. *Let C be a compact convex subset of a Banach space X. Then a continuous random operator* $f:\Omega\times C\to C$ *has a random fixed point.*

First we give a random approximation result and then we give solution to the nonlinear eigenvalue problem.

Theorem 6. *Let* $f:\Omega\times B_r\to X$ *be a continuous random operator and* $\mu(\omega)$ *any random constant. (*B_r *is a closed ball in a finite dimensional Banach space X). Then there exists an* $x(w)\in B_r$ *such that*

$$\|x(w)-f(w)\|=d(f(w),B_r).$$

Proof : Let $R:X\to B_r$ be a radial retraction. Then $g=R\,o\,f:B_r\to B_r$ is a continuous random operator and has a fixed point by Theorem 5 say $x(w)=R\,o\,f(w)$. Then as in Theorem 1 we get that

$$\|x(w)-f(w)\|=d(f(w),B_r).$$

We derive the following result as a corollary.

Let B_r be a ball in a finite dimensional Banach space X and $f:\Omega\times B_r\to B_r$.Then f has a random fixed point. In this case

$$\|x(w)-f(w\|=d(f(w),B_r)=0,$$

since $f(w)\in B_r$ for all $x\in B_r$. So $f(w)=x(w)$.

Now we give the following for nonlinear eigenvalue problem.

Theorem 7. *Let* $f,g:\Omega\times B_r\to X$ *be continuous operators and* $\mu(\omega)$ *any random constant. Then there exists a random solution* $x(w)\in B_r$ *of the equation*

$$f(w)\,x\,(w)=\mu(w)\,g(w)\,x(w)$$

provided $f(w)$ *and* $g(w)$ *satisfy*

(*) if for some $x\in\partial B_r$ *the equation*

$$f(w)x=\alpha x+\mu(w)g(w)\,x \text{ holds then } \alpha\le 0.$$

Proof : Let $h(w)x(w)=f(w)x(w)+x(w)-\mu(w)g(w)x(w)$. Then by Theorem 4 there is an $x(w)\in B_r$ such that

$$\|x(w)-h(w)x(w))\|=d(h(w),B_r).$$

If $x(w)\neq h(w)x(w)$ and $\|x(w)\|=r$ then $\|h(w)x(w)\|\ge r$. But then

$$f(w)x(w)-\mu(w)g(w)x(w)=(\alpha+1)x(w),$$

that is,

$$\frac{\|f(w)x(w) - \mu(w)g(w)x(w)\|}{\|x(w)\|} = \alpha + 1 > 1,$$

a contradiction to the hypothesis that $\alpha \leq 0$. Hence,

$$x(w) = f(w)x(w) + x(w) - \mu(w)g(w)x(w),$$

$$\text{i.e. } f(\omega)x(w) = \mu(w)g(w)x(w).$$

In case $g = I$, an identity function, then we have the following.

Corollary 3 : Let $f : \Omega \times B_r \to X$ be a continuous random operator and $\mu(w)$ a random constant. Then there is a random solution $x(w) \in B_r$ of the equation $f(w)x(w) = \mu(w)x(w)$ provided the following holds : if for some $x \in \partial B_r$ the equation

$f(w)x(w) = \alpha x(w) + \mu(w)x(w)$ holds then $\alpha \leq 0$.

Proof : By Theorem 4 we have

$$\|x(w) - h(w)x(w)\| = d(h(w), B_r)$$

where $h(w)x(w) = f(w)x(w) - (\mu - 1)x(w)$.

If $x \in \partial B_r$ and $f(w)x(w) \neq \mu(w)x(w)$ then $\|x(w)\| = r$ and $\|h(w)x(w)\| \geq r$.

So $h(w)x(w) = f(w)x(w) - (\mu - 1)x(w) = (\alpha + 1)x(w)$.

Hence,

$$\frac{\|h(w)x(w)\|}{\|x(w)\|} = \alpha + 1 > 1$$

contradicting the fact that $\alpha \leq 0$ so $f(w)x(w) = \mu(w)x(w)$.

If $f : X \to X$ is a continuous function and X is a Banach space. The function f is said to be densifying if $\alpha(f(A)) < \alpha(A)$ for each bounded set $A \subset X$ and $\alpha(A) > 0$. Here α is the measure of noncompactness defined as

$$\alpha(A) = \inf \{ \in > 0 \text{ such that } A \text{ is covered by a finite number of sets of diameter} \leq \in \}.$$

If A is totally bounded then $\alpha(A) = 0$.

If $\alpha(f(A)) \leq \alpha(A)$ then f is said to be 1-set contraction.

Recently Shan [8] gave the following.

Theorem 8. *Let $f : B_r \to X$ be a 1-set contraction map. Suppose f satisfies the following :*

**if $\{x_n\}$ is any sequence in B_r such that $x_n - fx_n \to 0$ as $n \to \infty$ then there exists a $y \in B_r$ with $y - fy = 0$.*

Then there is a $y \in B_r$ such that

$$\|y - fy\| = d(fy, B_r).$$

This can be further extended to cover inward/outward map *and fixed points.*

Theorem 9. *Let $f: B_r \to X$ be a 1-set contraction map satisfying the following :*

if $\{x_n\}$ is any sequence in B_r such that $x_n - fx_n \to 0$ as $n \to \infty$ then there exists a $y \in B_r$ with $y - fy = 0$. Then either f has a fixed point or there exists a $y \in B_r$ such that

$$0 < \|y - fy\| = d(fy, \overline{I_{B_r}(y)}).$$

Proof : By Theorem 8 we have that there is a $y \in B_r$ such that $\|y - fy\| = d(fy, B_r)$.

If $\|y - fy\| = 0$ then y is a fixed point.

Suppose $0 < \|fy - y\|$. Then

$$\|y - fy\| \leq \|fy - z\| \text{ for all } z \in I_{B_r}(y).$$

This follows on the same lines as of Theorem 4.

It is easy to derive fixed point theorems for inward maps and outward maps.

Note : If $f: B_r \to X$ with the hypothesis of Theorem 9, an aditional condition satisfies :

if $x \in B_r$ then $\|fx - x\|^2 \geq \|fx\|^2 - \|x\|^2$,

then f has a fixed point.

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