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SOME COUNTEREXAMPLES IN FIXED POINT THEORY

by

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ABSTRACT

We give a few counterexamples in Fixed Point Theory for multivalued mappings.

In this short paper we present a few counterexamples of various nature in the area of Fixed Point Theory for multivalued mappings.

They are not directed to an audience of experts in the field, but they are devoted to an audience of neophytes of Fixed Point Theory; neophytes are above all rich of little experience.

Although the counterexamples presented here are very elementary, we think they can be more useful than laborious but "fancy" fixed point Theorems.

In the following we will label :

– H a Hilbert space.

– E a Banach space with norm $\| \ \|$.

– X a metric space with distance d.

– 2^X the non empty subsets of X.

– K a subset of X.

– $CB(X)$ the family of nonempty bounded closed subsets of X.

– $KC(E)$ the family of nonempty compact convex subsets of E.

– For $A, B \in CB(X)$, $D(A, B) := \max\,(\sup_{b \in B} d(b, A), (\sup_{b \in A} d(a, B))$,

where $d(X, Y) := \inf_{y \in Y} d(x, y)$, is the Hausdorff distance from A to B.

– f a single-value map from K to X.

– T a multivalued map from K to 2^X.

– $Fix(T)$ the set of fixed points of T, $Fix(T) := \{z \in K : z \in Tz\}$.

– For $A, B \in CB(X)$, $\delta(A, B) := \sup \{d(a, b) :$
$a \in A, b \in B\}$, $\delta(x, A) := \delta(\{x\}, A)$.

COUNTEREXAMPLES 1, 2 AND 3.

In [1] Caristi states (and it could not be said better): "An inwardness condition is one which assert that, in some sense, points from the domain are mapped "toward" the domain . . . Possibly the weakest of the inwardness conditions, the Leray- Schauder boundary condition, is the assumption that T maps points x of ∂K anywhere except to the outward part of the ray originating at some interior point of K and passing through x, i.e.

$\exists\, w \in K^0$ such that " $x \in \partial K,\ f(x) - w = m(x - w)$ implies $m \leq 1$. ... (1)

If $x \in K$ we define the inward set, $I_K(x)$, as follows :

$$I_K(x) := \{x + c(u - x) \in E : u \in K \text{ and } c \geq 1\}.$$

A mapping $f : K \to E$ is said to be inward if $f(x) \in I_K(x)\ \ \forall\, x \in K$.

We say that f is weakly inward in case $f(x) \in \overline{I_K(x)}\ \ \forall\ x \in K$.

The proof of the fact that the Leray-Schauder condtion is weaker than the inwardness condition defined above, can be found for instance in [2] in the case K is convex.

The following counterexample shows that the converse is not true :

COUNTEREXAMPLE 1. Let $E = R^2$ with the euclidean norm

$$K := \{(a, b) \in E : a^2 + b^2 \leq 1\},\ f(a, b) := (a, b + |a|).$$

We show that f satisfies (1) with $w = 0$, i.e., $f(x) = mx$ implies $m \leq 1$.

Let $x = (a, b) \in \partial K$. Then $x = (a, \pm (1 - a^2)^{1/2})$ and $f(x) = (a, \pm (1 - a^2)^{1/2} + |a|)$.

Moreover let $f(x) = mx$, that is

$$\begin{cases} a = ma & (2) \\ \pm (1 - a^2)^{1/2} + |a| = m(\pm (1 - a^2)^{1/2}) & (3) \end{cases}$$

Now, if $a \neq 0$, furnish $m = 1$ and then (3) yields $a = 0$, a contradiction ! So $f(x) = mx$ only if $x = (0, \pm 1)$ and in this case (3) implies $m = 1$.

On the other had, one can see immediately that f is not weakly inward.

Some extensions of the (Weak) inwardness and Leray-Schauder condtions are useful also in the search for fixed points for multivalued mappings

(Results and references can be found for example in Deimling [2], [3]).

For example, if $T : K \to 2^E$ is a multivalued mapping, a suitable reformulation of (1) is

$\exists\, w \in K^0$ such that $\forall\, x \in \partial K,\ w + m(x - w) \in Tx$ implies $m \leq 1$ ... (4)
and a suitable reformulation of inwardness is

$$T\,x \cap I_K(x) \neq \phi \quad \forall\, x \in K. \qquad ...\ (5)$$

However, while in the univoque case (5) implies (4) whenever K is convex, but in the multivalued case this is no longer true :

COUNTEREXAMPLE 2.

Take E, K as in counterexample 1.

$$T(a, b) := (a, [b, b + 1]).$$

Then $(a, b) \in T(a, b)$ for $(a, b) \in K$, so T satisfies (5). We show that T does not satisfy (4).

Let $w \in K^0$, $w = (a_0, b_0)$. Let $x = x(w) = (a_0, (1 - a_0^2)^{1/2}) \in \partial k$, $z = (a_0,$
$(1 - a_0^2)^{1/2} + 1)$, $m = 1 + ((1 - a_0^2)^{1/2} - b_0)^{-1} > 1$. Then
$w + m(x - w) \in Tx$ and this means (4) is not satisfied.

REMARK. We observe that the map of Counterexample 2 verifies a wak Leray-Schauder condtion : $\exists\, w \in K^0$ such that $\forall\, x \in \partial K$, $w + m(x - w) \in Tx$ implies $\exists\, c \leq 1$ such that $w + c(x - w) \in Tx$. The last condtion was introduced by De Pascale-Guzzardi in [4] and in our opinion it is the true formulation of the Leray-Schauder condtion in the context of multivalued maps with convex values.

A similar situation occurs for nonexpansive and pseudocontractive maps : it is immediate to see that, in the univoque case, nonexpansivity implies pseudocountractivity, i.e. $\|f(x) - f(y)\| \leq \|x - y\|$ implies

$$\|x - y\| \leq \|(1 + r)\,(x - y) - r(f(x) - f(y))\| \quad \forall\, x, y \in K, r > 0.$$

In the multivalued case it is no longer true :

COUNTEREXAMPLE 3.

Let $E = R$, $K = E$, $Tx := [x, x + 1]$. T is nonexpansive (i.e. $D(Tx, Ty) \leq \|x - y\|$ but not pseudocontractive (i.e.
$\|x - y\| \leq \|(1 + r)\,(x - y) - r(u - v)\|$
$\forall\, x, y \in E,\ u \in Tx,\ v \in Ty, r > 0$ is not satisfied).

COUNTEREXAMPLES 4 AND 5

In the last years, the interest in optimization theory for the multivalued maps T satisfying Fix $(T) = \{z\}$ and $\{z\} = Tz$, has prompted a corresponding interest in Fixed Point Theory, since in [5] it has been shown that the maximization of a multivalued map T with respect to a cone, which subsumes ordinary and Pareto optimization, is equivalent to a fixed point problem of determining z such that $Tz = \{z\}$.

With this purpose, a sufficient condition that ensures $Fix(T) = \{z\}$ and $Tz = \{z\}$ for a multivalued map $T : X \to CB(X)$ is, for example

$$\delta(Tx, Ty) \le q \max\{d(x,y), \delta(x, Tx), \delta(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}$$
with $0 < q < 1$ ([6]).

On the other hand, if the multivalued mapping T satisfies

$$\delta(Tx, Ty) \le Ad(x, Tx) + Bd(y, Ty) + cd(x,y) \qquad \dots (6)$$

with $A, B \in]\,0, \infty[$ and $c \in]\,0, 1[$

or

$$\delta(Tx, Ty) \le a\delta(x, Tx) + b\delta(y, Ty) + cd(x,y) \qquad \dots (7)$$

then $Fix(T) \ne \phi$ obviously implies $Fix(T) = \{z\}$ and $Tz = \{z\}$.

A sufficient condition which ensures $Fix(T) \ne \phi$ is, for example

$$D(Tx, Ty) \le q \max\{d(x,y), \tfrac{1}{2}[d(x, Tx) + d(y, Ty)], \tfrac{1}{2}[d(x, Ty) + d(y, Tx)]\}$$

with $0 < q < 1$ [7]. ... (8)

However we note that the map $T : N \to N$ defined by $T0 = T1 = 0$, $Tn := \{0, 1\}, n \ge 2$, satisfies both (6) and (7) with $A = 10$, $B = 10, a = 0.4, b = 0.5, c = 0.9$ but it does not satisfy (8), although Fix $(T) \ne 0$. one could suspect that the condition (6) or (7) always implies $Fix\ (T) \ne \phi$. This is not true, as the following example shows :

COUNTEREXAMPLE 4.

Let $X = N$, $T0 := 1$, $Tn := \{0, 1\}$ for $n \ge 2$. Then (6) and (7) both are satisfied with $A = 20$, $B = 20$, $a = 0.4$, $b = 0.5$, $c = 0.9$. Nevertheless Fix $(T) = \phi$.

At this point, we recall that interesting results are known for the weak contraction single valued mappings (i.e. $d(f(x), f(y)) < d(x,y)$). Such contractions were studied first by Edelstein in [8].

In the case of a multivalued map T, taken into account that (6) and (7) are not sufficient to guarantee $Fix\ (T) \ne \phi$, not even if the domain of T is a compact set, one can suspect that the following condition is sufficiently strong :

$$\delta(Tx, Ty) < d(x,y) \text{ for } x \ne y. \qquad \dots (9)$$

Actually, such a condition is too strong, in the sense that nearly always do not exist "authentic" multivalued maps that satisfy (9) as shown in the following lemma :

LEMMA Let X be a metric space without isolated points. Let $T : X \to 2^X$ be a multivalued mapping satisfying (9). Then $T = f$ is a single valued map.

PROOF. Let $x \in X$ and $y_n \in B(x, 1/n)$, $y_n \ne x$. Then (9) implies

$$\delta(Ty_n, Tx) \to 0 \text{ for } n \to \infty \qquad \dots (10)$$

and from this its follows that Tx is a singleton. Indeed, if there exist $z, w \in Tx$ with $z \ne w$, we put $r = \frac{1}{2} d(z, w)$ and we show that

$$\delta(Ty_n, Tx) \ge r \text{ for each } n \qquad \dots (11)$$

contradicting (10). If fact, if $\forall\, v \in Ty_n$ it results $d(z, v) \geq r$, then (11) is obviously true. On the contrary, if there exists $v \in Ty_n$ such that $d(z, v) < r$ then $2r = d(z, w) \leq d(z, v) + d(v, w) < r + d(v, w)$, i.e. $d(v, w) > r$ and this still yields (11) since $(Ty_n, Tx) \geq d(v, w)$.

On the contrary, if the metric space X has isolated points, there exist multivalued mappings T satisfying (9) and in such a case $Fix(T)$ is a singleton ($Fix(T) = \{z\}$) of course. But that which is not verified in general is $Tz = \{z\}$ as the following counterexample shows :

COUNTEREXAMPLE 5.

$$X = \{0, 1, 2/3\} \cup \{2/3 - \sum_{k=1}^{n} 10^{-k},\ n \geq 1\}.\ T : X \to 2^X,\ T0 := \{0, 1\},$$

$$T1 := 2/3) := 2/3 - 10^{-1},\ T(2/3 - \sum_{k=1}^{n} 10^{-k}) = 2/3 - \sum_{k=1}^{n+1} 10^{-k}.$$

Then T satisfies (7), $Fix(T) = \{0\}$ but $T0 \neq \{0\}$.

COUNTEREXAMPLE 6

Finally, let H be a Hilbert space and K a closed convex subset of H. For a map $T : K \to KC(H)$ we define $\forall\, x \in K$

$$\hat{T}x := \{y \in Tx : d(y, K) = d(Tx, K)\},$$

the Fan's best approximation from Tx to K.

One can see that the convexity of Tx implies the convexity of $\hat{T}x$ and $P_K\hat{T}x$, is projection on K[9].

But without the assumption of convexity of Tx, the reader is invited to find examples of mappings such that T is nonexpansive while $\hat{T}$ is not.

In [8] was an open question if the nonexpansivity of $T : K \to KC(H)$ implies the nonexpansivity of $\hat{T}$. The following counterexample, due to *H.K.Xu*, negatively answers to such question :

COUNTEREXAMPLE 6.

$H = R^2$ with the euclidean norm. $K =$ triangle with vertices $D(\frac{1}{2}, 0), A(1, 0)$ and $C(\frac{1}{2}, \frac{1}{2})$. Define $T : K \to KC(H)$ as follows. Let $z = (x, y)$ be any point in K and let z' be the symmetric point of z with respect to the segment CD. Let P be the projection of z' onto the x-axis. Then we define

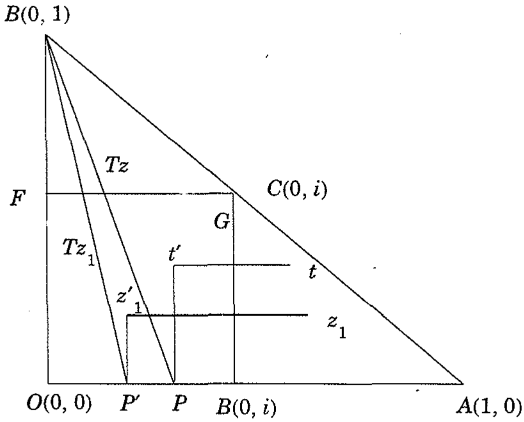
$Tz :=$ the segment BP linking B and P.

Then if $z = (x, y)$ and $z_1 = (x_1, y_1)$, we have $D(Tz, Tz_1) = |P - P'| \leq \|z - z_1\|$,

i.e. T is nonexpansive. It is also easily seen that

$$Tz = \begin{cases} P & \text{if } z \neq A \\ OF & \text{if } z = A. \end{cases}$$

It follows that for G on the open segment CD,



$D(\hat{T}G, \hat{T}A) = \sup\limits_{z \in 0F} d(z, D) = d(F, D) = d(C, A), d(G, A)$, so T is not nonexpansive.

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AN INTRODUCTION TO THE LIE-SANTILLI ISOTOPIC THEORY

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ABSTRACT

Lie's theory in its current formulation is linear, local and canonical. As such, it is not applicable to a growing number of nonlinear, nonlocal and noncanonical systems which have recently emerged in particle physics, superconductivity, astrophysics and other fields. In this paper, which is written by a physicist for mathematicians, we review and develop a generalization of Lie's theory proposed by the Italian-American physicst R.M. Santilli back in 1978 when at the Department of Mathematics of Harvard University and today called *Lie-Santilli isotheory*. The latter theory is based on the so-called isotopies which are nonlinear, nonlocal and noncanonical maps of any given linear, local and canonical theory capable of reconstructing linearity, locality and canonicity in certain generalized spaces and fields. The emerging Lie-Santilli isotheory is remarkable because it preserves the abstract axioms of Lie's theory while being applicable to nonlinear, nonlocal and noncanonical systems. After reviewing the foundations of the Lie-Santilli isoalgebras and isogroups, and introducing seemingly novel advances in their interconnections, we show that the Lie-santilli isotheory provides the invariance of all infinitely possible, non linear, nonlocal and noncanonical deformations of conventional Euclidean, Minkowskian or Riemannian invariants. We also show that the nonlinear, non local and noncanonical symmetry transformations of deformed generalized invariants are easily computable from the linear, local and canonical symmetry transforms of the original invariants and the given deformations. We then briefly indicate a number of applications of the isotheory in various fields. Numerous rather intriguing and open mathematical problems are indicated during the course of the presentation.

1. INTRODUCTION

1. A. Limitations of Lie's theory. As it is well known, *Lie's theory* has permitted outstanding achievements in various disciplines. Nevertheless, in its current conception [30] and realization (see, e.g.,

[15]), Lie's theory is *linear, local- differential and canonical-Hamiltonian.* As such, it possesses clear limitations.

An illustration is provided by the historical distinction introduced by Lagrange [29], Hamilton [14] and others between the *exterior dynamical problems* in vacuum and the *interior effectively* approximated as being point-like while moving within the homogeneous and isotropic vacuum under action-at-a-distance interactions (such as a space-ship in a stationary orbit around Earth). The point-like character of particles permits the validity of conventional local-diffrential topologies (i.e., the Zeeman topology in special relativity); the homogeneity and isotropy of space then allow the exact validity of the geometries underlying Lie's theory (such as the Riemannian geometry); and the action-at-a-distance interactions assures their representation via a potential with consequential canonical character.

Interrior problems consists of extended, and therefore deformable particles moving within inhomogeneous and anisotropic physical media, with action-at-a-distance as well as contactresistive interactions (such as a space-ship during re-entry in Earth's atomosphere). In the latter case the forces are of local- differential type (e.g., potential forces acting on the center- of-mass of the particle) as well as of nonlocal integral type (e.g., requiring an integral over the surface of the body), thus rendering inapplicable conventional local-differential topologies; the inhomogeneity and anisotropy of the medium imply the inapplicability of conventional geometries for their quantitative treatment; while contact- resistive interactions violate Helmholz's conditions for the existence of a potential (the *conditions of variantional selfadjointness* [49]), thus implying the noncanonical character of interior systems.

We can therefore say that Lie's theory in its conventional linear, local and canonical formulation is *exactly valid* for all exterior dynamical problems, while it is inapplicable (ant not "violated") for the more general interior dynamical problems on topological, geometrical, analytic and other grounds.

1. B. The need for a suitable generalizaion of Lie's theory. Lie's theory is currently applied to nonlinear, nonlocal and noncanonical systems via their simplifications into more treatable forms, e.g., via the expansion of nonlocal-integral terms into power series in the velocities and then the transformation of the system into a coordinate frame in which it admit a Hamiltonian via the Lie-Koening Theorem [49].

At times, however, nonlinear, nonlocal and nonhamiltonian systems cannot be consitently reduced or transformed into linear, local and Hamiltonian ones. An illustration exists in gravitation. The distinction between exterior and interior gravitational problems was in full use in the early part of this century (see, e.g., Schwartzschild's two papers, the first celebrated paper [72] on the exterior problem and the second little known paper [73] on the interior problem). The distinction was then kept in early well written treatises in the field (see, e.g., [4], [38]). The distinction was then progressively abandoned up to the current treatment of all gravitational problems, whether interior or exterior, via the same local-differential Riemannian geometry.

The above trend was based on the belief that interior dynamical problems within physical media can be effectively reduced to a collection of exterior problems in vacuum (e.g., the reduction of a space-ship during re-entry in our atmosphere to its elementary constituents moving in vacuum).

It is important for this paper to know that the above reduction is mathematically impossible. For instance, the so-called *No-Reduction Theorems* [54] prohibit the reduction of a microscopic interior system (such as satellite during re-entry) *with a monotonically decreasing angular momentum*, to a finite collection of elementary particles each one with *a conserved angular momentum*, and viceversa.

On geometrical grounds, gravitational collapse and other interior gravitational problems are not composed of ideal points, but instead of a large number of extended and hyperdense particles (such as protons, neutrons and other particles) in conditions of total mutual penetration, as well as of compression in large numbers into small regions of space. This implies the emergence of a structure which is arbitrarily nonlinear (in coordinates and velocities), nonlocal-integral (in various quantities) and non- hamiltonian (variationally nonselfadjoint).

Additional insufficiencies of the current formulation of Lie's theory and of its underlying geometries exist for the characterization of antimatter, e.g., because of the lack of a suitable (e.g., antiautomorphic) map which permits the characterization of antimatter, first, at the classical astrophysical level, and then at the level of its elementary constituents.

Similar occurences have recently emerged in astrophysics, superconductivity, theoretical biology and other disciplines. These occurrences establish the need for a generalization of the conventional Lie theory which is directly applicable (i.e., applicable without approximation or transformations) to nonlinear, integro-differential and variationally nonselfadjoint equations for the characterization of matter, and then possesses a suitable antiautomorphic map for the effective characteriszation of antimatter.

1.C. Santilli's isotopies of Lie's theory. In a seminal memoir [47] written in 1978 when at the Department of Mathematics of Harvard University under support from the U.S. Department of Energy, the Italian-American scholar Ruggero Maria Santilli proposed a step-by-step generalization of the conventional Lie theory specifically conceived for nonlinear, integro-differential and noncanonical equations. The generalized theory was subsequently studied by Santilli in ref.s ([48]-[71]), as well as by a number of a mathematicians and theoreticians, and it is today called Lie-Santilli isotopic theory or isotheory (see papers [1], [2], [8], [11], [12], [16]-[23], [25], [32], [33],

[35]-[37], [40]-[43], monographs [3], [24], [31], [74] and additional references quoted therein).

A main characteristic of the Lie-Santilli isotheory, which distinguishes it from all other possible generalizations, is its "isotopic" character intended (from the Greek meaning of the word) as the capability of preserving the original Lie axioms. More specifically, Santilli's isotopies are maps of any given linear, local and canonical structure into its most general possible nonlinear, nonlocal and noncanonical forms which are capable of reconstructing linearity, locality and canonicity in certain generalized isospaces and isofields within a fixed system of local coordinates.

The latter property is remarkable, mathematically and physically, in as much as it permits the preservation of the abstract Lie theory and the transition from exterior to interior problems via a more general *realization* of the same theory.

Another main characteristic of the Lie-Santilli isotheory is that of admitting a novel antiautomorphic map, called *isoduality*, which has resulted to be effective for the characterization of antimatter at the classical as well as operator levels.

It should be indicated that Santilli [47] submitted his isotopic theory as a particular case of a yet more general theory today called *Santilli's lie-admissble theory or Lie-Santilli genotopic theory* where the term genotopic is used (in its Greek meaning) to "induce configuration", and interpreted in the sense of violating the original Lie axioms, but inducing covering Lie-admissible axioms.

This paper is written by a theoretical physicist for mathematicians and it is soley devoted to the Lie-Santilli isotheory. A study of the broader Lie-Santilli genotheory is contemplated as a future work. In Sect. 2 we outline the methodological foundations of the theory. The isotopies of Lie's theory are presented in Sect. 3 jointly with new developments, such as a study of the transition from the Lie-Santilli isogroups to the corresponding isoalgebras. As an illustration of the capabilities of the isotheory, we prove its "direct universality" in gravitation, that is, the achievement of the symmetries of all possible gravitational metrics (universality), directly in the frame of the experimenter (direct universality). A number of fundamental open mathematical problems will be identified during the course of our analysis.

A comprehensive mathematical presentation of the Lie-Santilli isotheory up to 1992 is available in monograph [74]. A historical perspective is available in monograph [31]. Recent mathematical studies on isomanifolds (today called *Tsagas isomanifolds*) have been conducted in ref. [75] which is a topological complement of the algebraic studies of this paper.

2. Isotopies and Isodualities of Numbers, Fields, Differential Calculus, Metric Spaces, Differential Geometries, Functional Analysis, Classical and Quantum Mechanics.

Lie's theory is the embodiment of the virtual entirety of contemporary mathematics by encompassing : the theory of numbers, differential and exterior calculus, vector and metric spaces functional analysis' and others. Santilli's isotopies of Lie's theory require the isotopic lifting of all these mathematical methods. In this section we shall indicate the basic isotopies and isodualities which are necessary for a correct formulation of the Lie-Santilli isotheory.

2.A. Isotopies and isodualities of the unit. The fundamental isotopies from which all others can be uniquely derived are the unit I of the current formulation of Lie's theory into a quantity $\hat{I}$ of the same dimension of I, but with unrestricted functional dependence in the local coordinates x, their derivatives of arbitary order $\dot{x}, \ddot{x}$, ...as well as any needed addtional quantity [47], [49b], [61a]

$$I \to \hat{I} = \hat{I}(x, \dot{x}, \ddot{x}, ...). \qquad ... (2.1)$$

The *isotopies* occur when $\hat{I}$ preserves all the topological characteristics of I, such as nowhere-degeneracy, real-valuedness and positive-definiteness.

Once the unit is generalized, there is the natural emergence of the map [52], [53], [61a]

$$\hat{I} \to \hat{I}^d = -\hat{I}, \qquad ... (2.2)$$

called *isoduality* which provides an antiautomorphic image of all formulations based on $\hat{I}$.

The above liftings were classified by this author [22] into :

Class I (generalized units that are smooth, bounded, nondegenerate, Hermitean and positive-definite, characterizing the isotopies properly speaking);

Class II (the same as class I although $\hat{I}$ is negative-definite, characterizing isodulaties)

Class III (the union of Class I and II)

Class IV (singular isounits); and

Class V (unrestricted generalized units, e.g., realized via discontinuous functions, distributions, lattices, etc.).

All isotopic structures identified below also admit the same classifcation which will be omitted for brevity. In this paper we shall generally study isotopies of Classes I and II, at times treated in a unified way via those of Class III whenever no ambiguity arises. Santilli's isotopies of Class IV and V are vastly unexplored at this writing.

2.B. Isotopies and isodualities of fields. Lie's theory is constructed over ordinary fields $F(a, +, \times)$ hereon assumed to be of characteristic zero (the fields of real $\mathcal{R}$, complex C and quaternionic numbers Q) with generic elements a, addition $a_1 + a_2$ multiplication $a_1a_2 := a_1 \times a_2$, additive unit $0, a + 0 = 0 + a \equiv a$, and multiplicative unit $I, a \times I = I \times a \equiv a, \forall\ a, a_1, a_2 \in F$.

The Lie-Santilli isotheory is based on a generalization of the very notion of numbers and, consequently of fields first introduced by Santilli at the *Conference on Differential Geometric Methods in Mathematical Physics* held in Clausthal, Germany in 1980. A first rudimentary treatment appeared in Santilli's joint paper with the (mathematician) H.C. Myung [39] of 1982. Comprehensive studies were then conducted by Santilli in the following years (see paper [59] for a mathematical presentation and monographs [61] for extensive physical applications).

Consider a Class I lifting of the unit I of $F, I \to \hat{I}$ with $\hat{I}$ being outside the original set, $\hat{I} \notin F$. In order for $\hat{I}$ to be the left and right unit of the new theory, it is necessary to lift the conventional associative multiplication ab into the so-called *isomultiplication* [47]

$$ab := a \times b \Rightarrow a * b := a \times T \times a = a\, T\, b,\; T = \text{fixed}. \qquad \text{... (2.3)}$$

where the quantity T is called the isotopic element. Whenever $\hat{I} = T^{-1}, \hat{I}$ is the correct left and right unit of the theory, $\hat{I} * a = a * \hat{I} = a,\ \forall\ a \in F$, in which case (only) $\hat{I}$ is called the *isounit*. In turn, the liftings $I \to \hat{I}$ and $\times \to *$, imply the generalization of fields into the Class I structure

$$\hat{F}_I = \{\hat{a}, +, *) \mid \hat{a} = a\hat{I};\, a = n, c, q \in F; \times \to * = \times\, T \times; \hat{I} = T^{-1}\}, \qquad \text{... (2.4)}$$

called *isofields*, with elements $\hat{a} \in \hat{F}$ called *isonumbers* [59].

All conventional operations among numbers are evidently generalized in the transition from numbers to isonumbers. In fact, we have

$$a + b \to \hat{a} + \hat{b} = (a + b)\,\hat{I};\quad a_1 \times a_2 \to \hat{a}_1 * \hat{a}_2 = \hat{a}_1\, T\, \hat{a}_2 = (\hat{a}_1\, a_2)\hat{I};$$

$$a^{-1} \to \hat{a}^{\hat{I}} = a^{-1}\hat{I};\quad a/b = c \to \hat{a}/\hat{b} = \hat{c},\ \hat{c} = c\hat{I};\ a^{1/2} \to \hat{a}^{1\hat{/}2} = a^{1/2}\,\hat{I}^{1/2};$$

etc. Thus, conventional squares $a^2 = aa$ have no meaning under isotopy and must be lifted into the *isosquare* $\hat{a}^{\hat{2}} = \hat{a} * \hat{a}$. The *isonorm* is

$$\hat{|}\hat{a}\hat{|} = (\bar{a}a)^{1/2}\,\hat{I} = |a|\,\hat{I} \in \hat{F}, \qquad \text{... (2.5)}$$

where $\bar{a}$ denotes the conventional conjugation in F and $|a|$ the conventional norm. note that the *isonorm is positive-definite* (for isofields of Class I), as a necessary condition for isotopies.

The isotopic character of the lifting $I \to \hat{I}$ is confirmed by the fact that the isounit $\hat{I}$ verifies all axioms of I,

$$\hat{I} * \hat{I} * \ldots * \hat{I} \equiv \hat{I},\quad \hat{I}/\hat{I} \equiv \hat{I},\quad \hat{I}^{1\hat{/}2} \equiv \hat{I},\quad \text{etc.}$$

The *isodual isofields* are the antihomomorphic image of $\hat{F}(\hat{a}, +, *)$ induced by the map $\hat{I} \to \hat{I}^d = -\hat{I}$ and are given by the Class II structures

$$\hat{F}^d_{II} = \{(\hat{a}^d, +, *^d) \mid \hat{a}^d = \hat{a}\hat{I}^d;\ a = n, c, q \in F;\ * \to *^d$$

$$= \times\, T^d \times,\ T^d = -T,\ \hat{I}^d = -\hat{I}\}, \qquad \text{... (2.6)}$$

in which the elements $\hat{a}^d = \bar{a}\hat{I}^d$ are called *isodual isonumbers*. For real numbers we have $n^d = -n$, for complex numbers we have $c^d = -\bar{c}$, where $\bar{c}$ is the ordinary complex conjugate, and for quaternions in matrix representation we have $q^d = -q\tau$, where τ is the Hermitean conjugate. Noe That the conjugation of a complex number is $(n+ixm)^d = n^d + i^{d\,x^d}\, m^d = -n + (-\bar{i})(-x)(-m) = -n + im$. The isodual isosum is given by $\hat{a}^d + \hat{b}^d = (\bar{a}+\bar{b})\hat{I}^d$, while for isodual isomultiplication, we have

$$\hat{a}^d *^d \hat{b}^d = \hat{a}^d\, T^d\, \hat{b}^d = -\hat{a}^d\, T\, \hat{b}^d = (\bar{a}\,\bar{b})\hat{I}^d.$$

An important property is that *the norm of isodual isofields is negative-definite* because it is characterized by

$$\hat{|}\hat{a}^d\hat{|}^d = |\bar{a}|\hat{I}^d = -\hat{|}\hat{a}\hat{|}. \qquad \text{... (2.7)}$$

The letter property has nontrivial implications. For instance, it implies that *physical quantities defined on an isodual isofield, such as time, energy, etc.., are negative-definite*. For these reasons, isodual theories provide a novel and interiguing characterization of antimatter (see later on in this section the equivalence of isoduality and charge conjugation) [61].

Note also that, as a *necessary condition for isotopies (isodualities) all isofields $\hat{F}_I(\hat{a}, +, *)$ (isodual isofields $\hat{F}^d_{II}(\hat{a}^d, +, *^d)$) are isomorphic (antiisomorphic) to the original field $F(a, +, *)$*. The reader should be aware that the distinction between real, complex and quaternionic numbers is lost under isotopies because all possible numbers are unified by the isoreals owing the freedom in a generalized unit [26].

Recall that the set of imaginary numbers does not constitute a field, evidently because not closed under the multiplication. On the contrary, Santill's isofields $\hat{F}$ with isounit $\hat{I} = i$ do indeed verify the axioms for a field as one can readily verify. Note that the imaginary unit is *isoselfdual*, i.e., invariant under isoduality $i^d = -\bar{i} \equiv i$.

We also recall [59] that the lifting $a \to \hat{a} = a\hat{I}$ is *necessary* for $\hat{F}_I(\hat{a}, +, *)$ the preserve the axioms of $F(a, +, *)$ whenver the isounit $\hat{I}$ is not an element of the original field. On the contrary, when $\hat{I} \in F(a, +, *)$ there is no need to lift the numbers and we shall write $\hat{F}_I(a, +, *)$. In physical applications, the isounit is generally outside the original field and actually possesses a nonlinear as well as integral dependence on the local variables and their derivatives. This implies that the "numbers" used in the Lie-Santilli isotheory generally have an *integral* structure.

As an example, the isounit used by Animalu [1] for the representation of the Cooper pair in superconductivity is given by

$$\hat{I} = Ie^{t\, N \int d^3 \times \psi^{\tau}(r)\, \phi(r)}, \qquad \text{... (2.8)}$$

where t represents time, N is a positive real constant, and ψ and ϕ are the wave functions of the two electrons of the Cooper pair. *Annimalu's isounit* (2.8) therefore represents the nonlocal- integral contributions due to the wave overlapping of the two electrons in the

Cooper pairs which contributions, since they are of contact type, are variationally nonself adjoint and cannot be represented with the Hamiltonian.

In particular, Animalu has shown that the lifting of the conventional Coulomb interactions characterized by isounit (2.8) produces an attraction among the *identical* electrons of the Cooper pair, as experimentally extablished in superconductivity. Note that when the overlapping of the wavepackets is no longer appreciable (e.g., at large mutual distances), the integral in the exponent of (2.8) is null and the isounit $\hat{I}$ recovers the conventional unit I.

Conventional fields $F(a, +, \times)$ are used for large distances among the electrons, while isofields $\hat{F}(\hat{a}, +, *)$ with isounit (2.8) are used when the wave-overlapping of the electrons is appreciable. Other examples of isounits will be provided later on.

We also recall Santilli's [59] still more general *genofields*, characterized first by an isotopy of conventional fields, and then by an ordering of the isomultiplications, one to the right $\hat{a} > \hat{b} = \hat{A} \times R \times \hat{b}$ and one to the left $\hat{a} < \hat{b} = \hat{a} \times S \times \hat{b}$, $\hat{a} > \hat{b} \neq \hat{a} < \hat{b}$, $R \neq S$. The important property is that all abstract axioms of a field are verified per each ordered isomultiplication thus yielding a generalization of isofields which is at the foundation of Santilli's Lie-admissible Theory.'

We finally recall Santilli's [59] still more general liftings characterized by the generalization of the sum $+$ and related additive unit 0, e.g., $+ \rightarrow \hat{+} = K+, \hat{0} = K \neq 0, K \in F(a \hat{+} b = a + K + b)$ called *pseudo isotopies*, which do not preserve the axioms of a field (e.g., closure under the distributive law is not verified under the conventional $\times$ or isotopic $*$ multiplication and the addition $\hat{+}$). Thus, *pseudoisofields are not fields. For these and other reasons (e.g., the general divergence of the exponentiation), physical applications are restricted to iso-and geno-fields, while the pseudoiso-and pseudogeno-fields* have a mere mathematical interest at this writing. The care needed in inspecting and appraising the Lie- Santilli isotheory can be pointed out from these introductory lines. In fact, familiar statements such as "two multiplied by two equals four" are correct for the conventional Lie theory, but they have no mathematical meaning for the Lie-Santilli isotheory because they lack the identification of the assumed unit and multiplication. In fact, for $\hat{I} = 3$, $\hat{2} * \hat{2} = 12$. Similarly, care must be expressed before claiming that a number is *prime* or not. Infact, Santilli [59] has shown that non-prime numbers can become prime under a proper selection of the unit.

Santilli's theory of isonumbers is today sufficiently well known, and includes the lifting of all conventional numbers (real, complex and quaternionic numbers, plus the isotopies of octonions [59]) into the following four classes used in this paper : (A) *ordinary numbers* with unit 1; (B) *isonumbers* with isounits of Class I, $\hat{I} > 0$; (C) *isodual* *numbers* with isodual unit $I^d = -1$; (D) *isodual isonumbers* with isodual isounits of Class II, $\hat{I}^d < 0$. In this paper we shall therefore have *four* different types of real numbers, complex numbers and quaternions, excluding generalizations of Classes IV and V.

Despite the above advances, studies on the isonumber theory are their initiation and so much remains to be done. To begin, the entire conventional number theory (including all familiar theorems on factorization etc.) can be subjected to an isotopic lifting of Class I. Moreover, we have the birth of new numbers without counterpart in the current number theory, such as the isonumbers of Class IV (with singular isounits) and Class V (with distributions or discountinuous functions as isounits). All the above lifting then admit antiautomorphic images under isoduality which are absent in the conventional number theory. In turn, all the preceding generalizations can be subjected to a further enlargement via the differentiation of the multiplications to the right and to the left, and then yet more general formulations via the multivalued hyperstructures.

One can begin to understand the vastity of the Lie-Santilli isotheory as compared to the conventional formulation of Lie's theory by nothing that the above hierarchy of fields implies a corresponding hierarchy of Lie-isotopic theories.

2.C. Isotopies and isodualities of the differential calculus. The next important mathematical discovery by Santilli is an axiom-preserving integro-differential generalization of the conventional local-differential calculus called *isodifferential calculus*, first presented in a systematic way with applications in monographs [61], although implicit in preceding works (e.g., [58]).

Consider a set of functions $f(x), g(x), ...,$ on an N-dimensional space $S(x, \mathcal{R})$ with local chart $x = \{x^k\}, k = 1, 2, ..., N,$ over the reals $\mathcal{R}(n, +, x)$. Let dx^k and $\partial_k = \partial/\partial x^k$ be the conventional differential and derivative on S, respectively.

Consider now a set of functions $f(x), g(x), ...,$ this time, on an N-dimensional isospace $\hat{S}(x, \mathcal{R})$ with the same local chart x but defined over the isofield $\hat{S}(x, \mathcal{R})$ with N-dimensional Class I isounit $\hat{I} = \hat{I}^t = \{\hat{I}^j_i\} = \{\hat{I}^i_j\} = T^{-1} = [(T^j_i)]^{-1}$ possessing a generally nonlinear-integral dependence on local quantities and their derivatives, $\hat{I}(x, \dot{x}, \ddot{x}, ...)$. *Santilli's isodifferential calculus* is characterized by the *isodifferential*

$$\hat{d}\, x^k = \hat{I}^k{}_i\, dx^i, \qquad \text{... (2.9)}$$

with corresponding *isoderivative*

$$\hat{\partial}_k = \frac{\hat{\partial}}{\hat{\partial} x^k} = T^i_k\, \frac{\partial_i}{\partial x^k} = T^i_k\, \partial_i, \qquad \text{... (2.10)}$$

under the condtion that all conventional operations and properties of the ordinary differential calculus are lifted into their isotopic form, e.g.,

$$\hat{d}f(x) = \hat{\partial}_k f * \hat{d}x^k = \partial_i f\, T^i_j\, dx^j, \quad \hat{d}^{\hat{2}} x = \hat{d} * \hat{d}x = \hat{I}\, d^2 x,$$

and so on.

A hidden condtion is that, starting with a set of functions over a field $\mathcal{R}(n +, \times)$ with unit $+1$, the operations of isodifferentiation and isoderivatives must preserve the original unit for consistency. This condition remains generally unidentified in the conventional calculus because the preservation of the unit follows from its constancy, $\partial_k I \equiv 0$. For the case of a generalized unit with the same functional dependence of the functions, the conditon of preservation of the unit must be added to the calculus to prevent the transition from the original set of functions defined with respect to $\hat{I}$ to a new set of functions defined over a new unit $\hat{I}'$.

As an example, the defintion of the isodifferential

$$\hat{d}^k_x = d(T^k{}_i\, x^i) = (d\hat{I}^k{}_i)\, x^i + \hat{I}^k{}_i\, dx^i = \hat{I}'^k{}_i\, dx^i,\ \hat{I}'^k{}_i = (\partial_i \hat{I}^k{}_m)\, x^m + \hat{I}^k{}_i$$

would imply the alteration of the isounit $\hat{I}$. In turn, the occurrence would have serious drawbacks in applications, such as lack of invariance of perturbative series.

Santilli's isodifferential calculus does verify the condition of preserving the basic isounit, although the question whether realizations (2.9) and (2.10) are unique has not been explored until now. Note also the mutual compatibility of isoforms (2.9) and (2.10).

The lifting of the integral calculus follows quite simply from the above isodifferential forms. We here limit ourselves to indicate that an *indefinite isointegral* defined as the operation inverse of the isodifferential is given by

$$\hat{\int} \hat{d}x = \int T\, \hat{I} dx = \int dx = x, \qquad \text{i.e.} \qquad \hat{\int} \hat{=} \int T. \qquad \text{... (2.11)}$$

Note that the isodifferential calculus is one of the simplest possible forms of *integro-differential calculus*, in the sense that each operation has a differential contributions characterized by d or ∂ and an integral component characterized by T or $\hat{I}$, respectively.

Despite its simplicity, the isodifferential calculus has far reaching mathematical and physical implications. Mathematically, it permits a step-by-step generalization of conventional local- differential geometries into covering integro-differential geometries. Physically, the isocalculus permits a generalization of classical and quantum mechanics as well as of their interconnecting map (quantization), as outlined below.

The *isodual isodifferential calculus* is the antiautomorphic image of the preceding one characterized by the isodual isotopic element $T^d = -T$ or isodual isounit $\hat{I}^d = -\hat{I}$.

The *genodifferential calculus* [61] occurs when the Hermiticity conditon on the isounit is relaxed, $\hat{I} \neq \hat{I}^{\dagger}$. As such, the operation of differentiation itself acquires a structural ordering, namely, we have two different genoderivatives $\hat{\partial}^{>} f(x)$ and $f(x)^{<} \hat{\partial}$ defined for the corresponding units $\hat{I}$ and $\hat{I}^{\dagger}$ which are naturally set to represent the "arrow of time". This indicates that the genodifferential calculus is significant to represent *irreversible processes*.

2.D. Isospaces, isogeometries and their isoduals. Santilli's third important discovery presented for the first time in paper [51] of 1983 is the isotopic lifting of conventional, N- dimensional, metric (or pseudometric) spaces and related geometries. Consider a metric space $S(x, g, \mathcal{R})$ with local coordinates x and (nowhere singular, real valued) metric g over the reals $\mathcal{R}(n, +, \times)$. Its infinite class of isotopic images over the isoreals, called isospaces (hereon assumed for simplicity to be of Class I) is given by

$$\hat{S}(x, \hat{g}, \hat{\mathcal{R}}) : \hat{g} = Tg, \ \ \hat{I} = T^{-1}, x^{\hat{2}} = (x^t \, \hat{g} \, x) \, \hat{I} \in \mathcal{R}(\hat{n} +, \hat{*}), \qquad .. (2.12)$$

where $\hat{g} = g$ is called the *isometric*. The above liftings are necessary for compatibility with the isotopies of the unit $I \to \hat{I}$, of the product $x \to *$ and of the field $\mathcal{R} \to \hat{\mathcal{R}}$ The *isodual isospaces* [51] are given by

$$\hat{S}^d{}_{II} (x, \hat{g}^d, \hat{\mathcal{R}}^d): \ \hat{g}^d = T^d \eta, \quad T^d = -T, \ x^{\hat{2}d} = (x^t \, \hat{g}^d \, x) \hat{I}^d \in \ \hat{\mathcal{R}}^d (\hat{n}^d, +, *^d), \qquad (2.13)$$

where the subindices I or II will be often dropped hereon for simplicity. Note that $x^{\hat{2}} \equiv x^{\hat{2}d}$ and this may be the reason why isodual isospaces remained un-noticed until 1983 [51]. Isospaces and their isodual are however inequivalent, e.g., because the norm of the former is positive-definite, while that of the latter is negative-definite. By recalling that antimatter has been indentified in the negative-definite solutions of Dirac's equation and was originally though to evolve backward in time, isodual isospaces are ideally set to represent antimatter.

Despite their simplicity, isospaces have far reaching implications. In fact, they imply that *the same abstract axioms of conventional spaces (such as the Euclidean, Minskowskian or Riemannian spaces) admit unrestricted functional dependence of the metric*. As an illustration, the conventional Riemannian metric $g(x) \in R(x, g, \mathcal{R})$ is believed to be restricted to the sole dependence on the local coordinates x. Santilli has shown that the same Riemannian axioms permit an unrestricted functional dependence of the metric $\hat{g}(x, \dot{x}, \ddot{x}, ...)$. While Riemanian spaces $R(x, g, \mathcal{R})$ are ideally suited for *exterior* gravitational problems, the *Riemann-Santilli isospaces* $\hat{R}(x, \hat{g}, \mathcal{R})$ are ideally suited for the treatment of interior gravitational problems with a nonlinearity in the velocities, integral structure and nonselfadjoint character (Sect. 1).

This remarkable result is due to the construction of the isopaces via the deformation of the metric $g \to \hat{g} = Tg$ while jointly lifting the original unit in the *inverse* amount, $I \to \hat{I} = T^{-1}$, under which isospaces $\hat{S}(x, \hat{g}, \hat{\mathcal{R}})$ (*isodual isospaces* $\hat{S}^d(x, \hat{g}^d, \hat{\mathcal{R}}^d)$) *are locally isomorphic to the original spaces* $S(x, g, R)$, $(S^d(x, g^d, \mathcal{R}^d))$.

Additional salient properties of isospaces are the *preservation of the original dimensionality* and the *preservation of the original basis (except for renormalization factors)* [61a].

Via the use of the isotopies of fields, differential calculus and vector spaces, Santilli's has constructed step-by-step, nonlocal- integral isotopies and isodualities of conventional geometries on metric (or pseudo-metric)spaces. Their most salient application is the geometrization of interior physical media, that is, the geometrization of the departures from empty space caused by matter.

Santilli's isogeometries most important for physical applications are (see[61a] for details): A) The *isoeuclidean geometry* of Class I on three-dimensional isospaces $\hat{E}(x, \hat{\delta}, \hat{\mathcal{R}}), \hat{\delta} = T\delta$, $\delta = \text{diag.}\ (1, 1, 1)$, over the isoreals $\hat{\mathcal{R}}(\hat{n}, +, *)$ with isounit which can always be diagonalized into the form

$$\hat{I} = \text{diag.}\ (b_1^{-2}, b_2^{-2}, b_3^{-2}) > 0,\ b_k = b_k(x, \dot{x}, \ddot{x}, \dots ..) > 0,\ k = 1, 2, 3.$$

In this case the isometric $\hat{\delta}$ has an arbitrary functional dependence on local coordinates and their derivatives $\hat{\delta}(x, \dot{x}, \ddot{x}, \dots)$,. Yet the geometry is *isoflat*, that is, it verifies the axioms of flatness in isospace while its projection in the original space $E(x, \delta, \mathcal{R})$ is evidently curved. An intriguing novel notion of the isoeuclidean geometry is the *isosphere of Cass I* $\hat{x}^2 = (x^t\, \hat{\delta} x)\hat{I} \in \hat{\mathcal{R}}$ which is perfect sphere in isoeuclidean space. Nevertheless, its projection in the original euclidean space is given by all infinitely possible ellipsoids $xb_1^2x + y_2b^2y + zb_3^2z =$ const. In fact, in isospace the original sphere with radius I is subjected to the deformations of its axes $I_k \to b_k^2$ while the corresponding units are deformed in the *inverse* amounts, $I_k \to b_k^{-2}$, thus preserving the perfectly spherical character. The *isosphere* of Class *III* unifies all compact and noncompact curves $\pm xb_1^2x \pm y_2b^2y \pm zb_3^2z \neq 0$ in isospace. The *isosphere of Class IV* uniflies all compact and noncompact surfaces plus all cones $\pm xb_1^2x \pm y_2b^2y \pm zb_3^2z = 0$. The *isosphere of Class V* is an additional novel notion of a sphere with arbitrary unit (e.g., a lattice).

B) *The isominkowskian geometry of Class I* on isospace $\hat{M}(x, \hat{\eta}, \hat{\mathcal{R}}), \hat{\eta} = T\eta$ over the isoreals with isounit

$$\hat{I} = \text{diag.}\ (b_1^{-2}, b_2^{-2}, b_3^{-2}, b_4^{2}) > 0,\ \ b_\mu(x, \dot{x}, \ddot{x}, \dots) > 0, \mu = 1, 2, 3, 4.$$

... (2.15)

which represents *locally varying speeds of light* $c = c_0 b_4 = c_0/n_4$ where c_0 is the speed of light in vacuum and n_4 is the local index of refraction. As such, the isominkowskian geometry is particularly suited for the representation of light propagating wihin inhomogeneous and anisotropic physical media such as our atmosphere. An important notion of the isominkowskian geometry is the *isolight cone of Class I*, which is a perfect cone in isominkowski space but, when projected in the conventional Minkowski space, represents all infinitely possible deformed light cones $xn_1^{-2}x + y_2 n^{-2} y + zn_3^{-2} z - tc_0 n_4^{-2} t = 0$. In fact, each axis of the original ligiht cone is deformed $I_\mu \rightarrow n_\mu^{-2}$, while the corresponding units are deformed of the inverse amount, $I_\mu \rightarrow n_\mu^2$, thus preserving the original perfect cone. The axiom-preserving character of the isotopies is such that the maximal causal speed of the Minkowski and isominkowski spaces coincide and are given by the speed of light in vacuum c_0.

C) The *isoriemannian geometry* on isospaces $\hat{R}(x, \hat{g}, \hat{R})$, $\hat{g} = Tg$ over isounit (2.14), which coincides with the conventional geometry at the abstract level. This implies that, unlike the isoeuclidean and isominkowskian geometries, the isoriemannian geometry is *isocurved*, that is curved in isospace. As such, it permits the representation of interior gravitational problesm with locally varying speeds of light such as the bending of light within a physical medium with local speed $c = c_0/n_4 < c_0$, the contribution due to cosmological redshift due to the decrease of the speed of light within astrophysical chromospheres, and others novel insights. An intriguing novel notion is that of isogeodesics of class I which coincide in isospace with the geodesics in vacuum, but when projected in the original Riemannian space represents the actual nongeodesic trajectory within physical media, such as that of a leaf in free fall in our atmosphere.

An isogeometry particularly important for the study of the Lie-Santili theory is the *isosymplectic geometry* first presented in ref. [57]. Consider the conventional symplectric geometry (see, e.g. [34]) in canonical realization on the cotangent bundle $T{*}E(r, \delta, T)$, with local chart $a = \{a^\mu\} = \{p_k, r^k\}$, $k = 1, 2, 3$, $\mu = 1, 2, 3, 4{,}5, 6$. As well known, the above geometry is characterized by the canonical one form

$$\theta = p_k\, dx^k, \qquad \text{... (2.16)}$$

and the cannocial symplectic two-form

$$w = dp_k \wedge dx^k = \tfrac{1}{2} w_{\mu\nu}\, da^\mu \wedge da^\nu,\ (w_{\mu\nu}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad \text{... (2.17)}$$

which is exact, $w = d\theta$ and therefore closed $dw = d(d\theta) \equiv 0$ (Poincare lemma [34]), with corresponding higher-order forms.

Santilli's isosymplectic geometry is defined on the *isocotangent bundle* $T * \hat{E}(r, \hat{\delta}, \hat{\mathcal{R}})$ with the same local chart $a = \{p, r\}$ abut now referred to the isounit $\hat{I}_2 = \mathrm{diag.}\,\{T, \hat{I}\}$, where T appears beacuse referred to the *covariant momentum* p_k. It is characterized by *the one-isoform*

$$\hat{\theta} = p_k\, \hat{d}x^k = \hat{I}^i_j\,(x, p, ..)\, p_i\, dx^j, \qquad ... (2.18)$$

and the *two-isoform*

$$\hat{w} = \hat{d}p_k \wedge \hat{d}x^k = \tfrac{1}{2}\, w_{\mu\nu}\, \hat{d}a^\mu \wedge da^\nu = T^m_k(, p, ...)\, dp_m \wedge \hat{I}^k_n(x, p, ...)\, dx^n \equiv w, ... \quad (2.19)$$

which is also *isoexact*, $\hat{w} = \hat{d}\hat{\theta}$, and therefore *isoclosed*, $\hat{d}\hat{w} \equiv 0$ (isotopic Poincare′ Lemma [57], thus *isosymplectic;* and higher- dimensional isoforms.

Note that $\hat{w} \equiv w$ and this shows the "hidden" character of the isotopies. This also confirms that the *symplectic and isosymplectic geometries coincide at the abstract, realization- free level,* as established by the absract identity of forms (2.15) and (2.18), or (2.17) and (2.19). However, *the two geometries admit inequivalent realizations.* In fact, the symplectic geometry is strictly local-differential, does not admit nonlinearities in the velocities and it is canonical. On the contrary, the isosymplectic geometry has an integro- differential structure (in the sense indicated earlier), it is arbitrarily nonlinear in the velocities and it is noncanonical.

Regrettably, we have to refer the interested reader to monographs [61], [62] and paper [75].

At this writing, the isogeometries are sufficiently well known for physical applications. Nevertheless, their mathematical study is just at the beginning and a number of fundamental aspects remain open for study.

2.E Isotopies and isodualities of functional analysis. As indicated earlier, the isotopies imply nontrivial generalizations of *all* mathematical structures of Lie's theory, inevitably leading to a generalization of functional analysis called by this author *functional isoanalysis* [22].

The generalized discipline begins with the isotopy of continuity (whose knowledge is assumed when dealing with technical aspects of Section 3), and inlcudes the isotopies of conventional square-integrable, Banach and Hilbert spaces, as well as the isotopies of all operations on them.

In particular, functional isoanalysis includes a generalization of conventional special functions, distribbutions and transforms. For instance, the conventional Dirac delta distribution has no meaning under isotopy, mathematically, because of the loss of applicability of the conventional exponentiation and, physically, because particles are no longer point-like. The *isodirace distribution* is the the reconstruction of the conventional distribution for an unrestricted unit permitting a direct treatment of the extended character of particles. The Fourier transform, Legendre polynomials, etc., also admit simple yet unique and unambiguous isotopies with important applications in various disciplines.

Regrettably, we are unable to review the above isotopies to prevent a prohibitive length of this paper, and refer the interested reader to [61a]. One should however be aware that the elaboration of the Lie-Santilli isotheory via conventional functional analysis (e.g., the use of conventional trigonometry, logarithms, exponentiations, etc.) leads to inconsistencies which often remain undetected by the noninitiated reader.

2.F: Isotopies and isodualities of classical mechanics. As well known (see, e.g., [13]), Lie's theory admits two fundamental realizations, one in classical and one in quantum mechanics, with interconnecting map given by naive or symplectic quantization.

The preceding isotopies of fields, differential calculus, metric spaces, geometries, and functional analysis were used by Santilli for the construction of step-by step isotopic generalization of classical [62] and quantum [61] mechanics and their interconnecting maps. The new mechanics have been specifically conceived for the most general possible, nonlinear, nonlocal and noncanonical, interior dynamical problems, as well as for the fundamental classical and operator realizations of the Lie- Santilli isotheory. As a matter of fact, Santilli proposed the isotopies of Lie's theory precisely for quantitative treatment of the above generalized mechanics.

It is important to review at least the essential structural elements of the isotopic classical and operator mechanics because they provide the most important realizations of the Lie-Santilli isotheory.

To begin their outline, the isodifferential calculus requires the lifting of both the *space and time isodifferentials*

$$\hat{d}x^k = {}_s\hat{I}^k{}_i\, dx^i, \quad \hat{d}t = {}_t\hat{I}\, dt, \quad \hat{d}x_k = T^i_k\, dx_i, \quad \hat{d}p_k = T^i_k\, dp_k, \quad \text{... (2.20)}$$

where $s^{\hat{I}} = ({}_sT)^{-1} = {}_t\hat{I} = ({}_tT)^{-1}$ are the *space and time isounits with corresponding space and time isotopic elements* ${}_sT$ and $\hat{I}_t = T_t^{-1}$. The corresponding *space and time isoderivatives* are given by

$$\partial_k = \hat{\partial}/\hat{\partial}\, x^k = T^i_k\, \partial_i, \quad \hat{\partial}_t = T_t\, \partial_t, \quad \hat{\partial}^k = \hat{\partial}/\hat{\partial}x_k = \hat{I}^k_i\, \partial^i. \quad \text{... (2.21)}$$

The application of the isodifferential calculus to classical Hamiltonian mechanics then implies the following generalizations :

1) Newton's equations of motion $md^2x_k/dt^2 = -\partial_k V + F_k^{NSA}$ on $E(x, \delta, \mathcal{R})$ over $\mathcal{R}(n, +, \times)$ are lifted into the Newton- Santilli equations on the isoeuclidean space $\hat{E}(x, \hat{\delta}, \hat{\mathcal{R}})$

$$\hat{m}\hat{d}^2\, x_k/\hat{d}t^2 = -\hat{\partial}_k\, V(x), \quad \text{... (2.22)}$$

which, when projected in the original space, read

$$mT_t^2 \frac{d^2({}_S T_k \hat{\delta}_{ij} x^j j)}{dt^2} = - {}_s T^i_k \frac{\partial V(\hat{x})}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x^i}; \qquad \text{... (2.23)}$$

2) the conventional canonical action $\int_{t_1}^{t_2} (p_k dx^k - Hdt)$ is lifted into the isoaction

$$\hat{A} = \int_{t_1}^{t_2} [p_k \hat{d}x^k - H\hat{d}t] = \int_{t_1}^{t_2} dt\, L(t, x, \dot{x}, \dot{x}, ...) =$$

$$= \int_{t_1}^{t_2} dt\, [p_{ks} \hat{I}^k_i (t, x, p, ...) dr^i - H_t\, T(t, x, p, ...)dt]; \qquad \text{... (2.24)}$$

3) Hamilton's equations without external terms are lifted into the *Hamilton-Santilli* equations

$$\frac{\hat{d}\, x^k}{\hat{d}t} = \frac{\hat{\partial}\hat{H}(t, x, p)}{\hat{\partial}p_k}, \frac{\hat{d}p_k}{\hat{d}t} = \frac{\hat{\partial}\hat{H}(t_1 x_1 p)}{\hat{\partial}\, x^k}, \qquad \text{... (2.25)}$$

4) The conventional Lagrange equations and underlying Legendre transform are lifted into the form

$$\frac{\hat{d}}{\hat{d}t} \frac{\hat{\partial}\hat{L}(t, x, \dot{x})}{\hat{\partial}\, \dot{x}^k} - \frac{\hat{\partial}\hat{L}(t, x, \dot{x})}{\hat{\partial}\, x^k} = 0, \; p_k = \frac{\hat{\partial}\hat{L}}{\partial \dot{x}^k}, \; \dot{x} = \hat{d}x/\hat{d}t, \qquad \text{... (2.26)}$$

5) The Hamilton-Jacobi equations are lifted into the *isotopic Hamilton-Jacobi equaitons*

$$\hat{\delta}_t \hat{A} + \hat{H} = 0, \quad \hat{\partial}/\hat{\partial}r^k - p_k = 0, \; \hat{\partial}\hat{A}/\hat{\partial}p_k \equiv 0; \qquad \text{... (2.27)}$$

6) The conventional Poisson brackets, which are the realization in classical mechanics (CM) of the Lie product, are lifted into the *isopoisson* brackets first introduced in ref. [47] (see also [57] for the explicit form below)

$$[A \hat{,} B]_{CM} = \frac{\hat{\partial}A}{\partial x^i} {}_s T^i_j \frac{\hat{\partial}B}{\hat{\partial}p_j} - \frac{\hat{\partial}B}{\hat{\partial}x^i} T^i_{sj} \frac{\hat{\partial}A}{\hat{\partial}p_j} = \frac{\partial A}{\partial x^i} {}_s T^i_j \frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial x^i} {}_s T^i_j \frac{\partial B}{\partial p_j}, \qquad \text{... (2.28)}$$

which provide the desired classical realization of the Lie- Santilli brackets. In fact, it is easy to prove that the above brackets satisfy in isospace (but not in the original space) the Lie algebra axioms (see ref.s [61], [62] for a proof via the isotopies of the Poincare′ lemma in the symplectic geometry).

(7) The classical realization of the exponentiated form of Hamilton's equation as a realization of a one parameter Lie transformation group is lifted into the exponentiated from of Eq.s (2.25)

$$a' = \{e^{tw^{\mu\nu} \hat{\partial}_\mu \hat{H} \hat{\partial}_\nu}\} * \times = e^{tw^{\mu\nu} T^\alpha_\nu \partial_\mu H \partial_\alpha\}}\, a, \, a = \{p, r\}, \qquad \text{... (2.29)}$$

which is a realization of the one-dimensional Lie-isotopic groups, as we shall see in the next section.

The emerging new mechanics is called *Hamilton-Santilli* isomechanics. Some of the advantages over the conventional Hamiltonian mechanics are the following :

1) In Newton's equations of motion the bodies are approximated as massive points, as well known since Newton's time, and the same approximation persists at in Hamiltonian mechanics. The Newton-Santilli equations can represent instead the actual nonspherical shape of the body considered as well as its possible deformations. As a simple illustration one can assume for isotopic element the matrix with constant elements ${}_sT = \text{diag.}\ (a^{-2}, b^{-2}, c^{-2}, a, b, c \neq 0,$ which represents in isoeuclidean space a spheroidal ellipsoid. More complex shapes can be represented with nondiagonal isotopic elements. Deformations of the original shape due to external fields can be represented with nondiagonal isotopic elements. Deformations of the original shape due to external fields can be represented via suitable dependence of the isotopic element. The same representation of shape evidently persists in the Hamilton-Santilli equations (2.25) [see [61] for details]).

2) Only some of Newton's equations of motion are representable with Hamilton's equations in the cordinates x of the observer because they are in general *essentially noneselfadjoint* [49b]. In this case a Hamiltonian representation requires certain coordinates transformations $x \rightarrow x'(t, x, p), p \rightarrow p'(t, x, p)$ which are nonlinear, and therefore not realizable in actual experiment. The Hamilton- Santilli equations do permit instead the representation of these systems in the actual coordinates x of the observer. In fact, Eq.s (2.25) are variationally selfadjoint in isospace but not in regular space. A such, Eq.s (2.25) achieve an analytic representation in isoeuclidean space of systems which do not admit one in the original Euclidean space (see also [61] for details).

3) Newton's equations for bodies moving within a resitive medium are integro-differential

$$m\ddot{x} = -\partial V + \int_{\sigma} d\sigma \;\; \mathcal{F}(\sigma, \ldots), \qquad \ldots (2.30)$$

namely, they are differential with respect to the trajectory x of the centre of mass, while admitting corrections due to the shape σ of the body where $\mathcal{F}$ is a suitable kernel. The representation of these systems via Hamilton's equations is prohibited because the toplogy of the underlying symplectic geometry is strictly local-differential. The Hamilton-Santilli equations (2.25) can instead represent the above integro- differential systems because the underlying isosymplectic geometry does indeed admit such systems.

2.G: Isotopies of quantum mechanics. We now outline the operator realization of the Lie-Santilli isotheory first identified in ref. [48] and then studied in numerous subsequent papers (see ref. [61b] for the most recent account).

The isotopies of quantum mechanics were originally proposed by Santilli [48] under the name of *hadronic mechanics*, namely, a mechanics built for strongly interacting particles called hadrons. Since the charge radius of hadrons is of the same order of magnitude of the range of the strong interactions, and hadrons are the densest objects measured in laboratory under now, the activation of the strong interactions requires the mutual penetratrion of these hyperdense particles. This implies the expectation of nonlinear, nonlocal-integral and nonhamiltonian contributions in the strong interactions resulting in the need for a suitable generalization of quantum mechanics.

Let ξ be the enveloping associative operator algebra of quantum mechanics with elements A, B, .. unit I and conventional associative product AB, and let $\mathcal{H}$ be the conventional Hilbert space with states $|\psi>$, $|\phi>$, ... and inner product $<\psi | \phi> = \int d^3 r \psi^\tau(t, r)\phi(t, r)$ over the complex numbers $C(c, +, \times)$.

Hadronic mechanics is based on the lifting of the (sapce) unit $I \rightarrow \hat{I} = T^{-1}$ with consequential lifting of ξ into the *enveloping isoassociative algebra* $\hat{\xi}$ with the same elements A, B, ... now equipped with the isoassociative product $A * B = A\hat{T}B$, and lifting of $\mathcal{H}$ into the *isohilbert space* $\hat{\mathcal{H}}$ with *isostates* $|\hat{\psi}>$, $|\hat{\phi}>$, ... and *isoinner product* over the isofield of complex numbers

$$\hat{\mathcal{H}} : <\hat{\phi} \uparrow \hat{\psi}> = <\hat{\phi} | T | \hat{\psi}> \hat{I} \in \hat{C}(\hat{c}, +, *). \qquad ... (2.31)$$

The fundamental dynamical equations of hadronic mechanics can be uniquely and unambigously derived from the Hamilton-Santilli isomechanics via the isotopies of conventional or symplectic quantization. Recall that the *naive quantization* can be expressed via the mapping of the canonical action

$$A = \int_{*}^{t_2} (pdx^k - H\, dt) \rightarrow -ih\, Ln\psi(t, r). \qquad ... (2.32)$$

Such a mapping is now inapplicable to isoaction (2.24) because $\hat{A} \neq A$. But $h = 1$ is the basic unit of quantum mechanics which is replaced under isotopies by the (space) isounit $\hat{I}$. The consistent application of the isotopies then yields the generalized mapping identified by Animalu and Santilli and here presented for simplicity for the isounit independent from the local time and coordinates (but dependent on the velocities as essential for contact resistive forces, see [61b] for the general case and references)

$$\hat{A} = \int_{t_1}^{t_2} [p_k \hat{d}r^k - H\hat{d}t] \rightarrow -i\hat{I}\, Ln\, \psi\, (t, r), \qquad ... (2.33)$$

characterizing the *naive isoquantization* of generalized Hamilton-Jacobi equations (2.27) into the following fundamental dynamical equations of hadronic mechanics (see ref. (61b) for all references and details): *the isoschrödinger equations for the linear momentum*

$$-i\hat{\partial}_k\psi(t,r) = -iT^i_k\,\partial_i\,\psi(t,r) = p_k * \psi(t,r) = p_k\,T\psi(t,r), \qquad \text{... (2.34)}$$

with the related *fundamental isocommutation rules*

$$p_u * r^j - r^j * p_i = -\hat{\delta}^j_i = -\hat{I}^j_i; \qquad \text{... (2.35)}$$

first identified by Santilli; the *isoscrödinger equation for the energy*

$$i\hat{\partial}_t\,\psi\,(t,r) = i_t T\,\partial t_t\,\hat{\psi}pt, r) = \hat{H} * \psi(t,r) = \hat{H}\,T_s\psi(t,r) = \hat{E} * \psi(t,r) = E\psi(t,r), \qquad \text{... (2.36)}$$

first identified by Myung and Santilli with the conventional differential calculus, and finalized by Santilli with the isoderivatives; and the *isoheisenberg equation*

$$i\hat{d}Q/\hat{d}t = [Q\hat{,}H] = Q * H - H * Q = QTH - HTQ. \qquad (2.37)$$

with integrated form

$$Q(t) = e^{iHTt}\,Q(0)e^{-itTH}, \qquad \text{... (2.38)}$$

first identified by Santilli in the original proposal to build hadronic mechanics [48].

It should be recalled for subsequent need that *the condition of isohermiticity on an isohilbert space coincide with the conventional Hermiticity* $\hat{X^\tau} = X^\tau$. As a *consequence, all opeartors which are Hermitean in quantum mechanics remain so in hadronic mechanics.*

Also, unitary transforms on $\mathcal{H}$, $U\,U^\tau = U^\tau = I$,are lifted under isotopies into the *isounitary transformstions*

$$\hat{U} * \hat{U}^\tau = \hat{U}^\tau * U = \hat{I}.$$

The *fundamental operator realization of the Lie-Santilli isoproduct* is therefore givenby $[A, B] = ATB - BTA$ where A, B are elements of the isoenvelope $\hat{\xi}$ on an isohilbert space $\hat{\mathcal{H}}$ over the isofield $\hat{C}(\hat{c}, +, *)$. The *fundamental operator realization of the isogroups* is then given by Eq. (2.38) which, as we shall see, can be rewritten in terms of the isounitary transforms.

The advantages of hadronic over quantum mechanics are similar to those of the Hamilton-Santilli over the Hamiltonian mechanics. In fact, quantum mechanics is local-differential-potential and, as such, can only represent (in first quantization) point particles under action-at-a-distance interactions. By comparison, hadronic mechanics can represent (in first isoquantization) the actual nonspherical shape of the particles, all their infinitely possible deformations as well as nonlocal-integral interactions due to mutual penetration of the wavepackets of hadrons. The possibilities for broader applications in various disciplines is then evident.

2.E: Isolinearity, isolocality and isocanonicity. In Sect. 1 we pointed out that the primary limitations of the contemporary formulation of Lie's theory for applications are those of being linear, local and canonical. The classical realizations identified earlier indicate rather clearly that the Lie-Santilli isotheory is nonlinear, nonlocal and noncanonical, as desired.

It is important to understand that such nonlinearity, nonlocality and noncanonicity occur only when the theory is projected in the original space over the original fields because the theory reconstructs linearity, locality and canonicity in isospace (see [61] for all details and references).

Let $S(x, F)$, be a conventional vector space with local coordinates x over a field F, and let $x' = A(w)x$ be a linear, local and canonical transformation on $S(x, F)$, $w \in F$. The lifting $S(x, F) \to \hat{S}(x, \hat{F})$ requires a corresponding necessary isotopy [47]

$$x' = \hat{A}(\hat{w}) * \hat{A}(\hat{w})Tx,\ T \text{ fixed},\ x \in \hat{S}(x, \hat{F}),\ \hat{w} \quad I = T^{-1}, \qquad \text{... (2.39)}$$

called *isotransforms*, with isodual $x' = \hat{A}^d(\hat{w}^d)*^d x = -\hat{A}(\hat{w})*x$.

It is easy to see that isotransforms satisfy the condition of linearity in isospaces, called by Santilli *isolinearity*

$$A * (\hat{a} * \times + \hat{b} * y) = \hat{a} * (A * x) + \hat{b}*(A * y),\ \forall\, x, y \in \hat{S}(x, \hat{F}),\quad \hat{a}, \hat{b} \in \hat{F}. \qquad \text{... (2.40)}$$

althouth their projection in the original space $S(x, F)$ the isotransforms are nonlinear, because $x' = \hat{A}T(x, \dot{x}, \ldots)x$. Isotransforms (2.39) are also *isolocal* because the theory formally deals with the local variables x while all nonlocal terms are embedded in the isounit, namely, all nonlocal-integral terms disappear at the abstract, realization-free level. Nevertheless, the theory is nonlocal when projected in the original space. Similarly, isotopic theories are *isocanonical* because they are from the isoaction (2.24) which coincides at the abstract level with the canonical action.

3. Isotopies and Isodualities of Enveloping Algebras, Lie Algebras, Lie Groups, Symmetries, Representation Theory and Their Applications

As recalled in Sect.1, Lie's theory (see, e.g.,[13], [15] and [76]) is centrally dependent on the basic n-dimensional unit $I=$ diag. (1, 1 ,..., 1) in all its major branches, such as enveloping algebras, Lie algebras, Lie groups, representation theory, etc. The main idea of the Lie-santilli theory [47], [49], [61], [62] is the reformulation of the entire conventional theory with respect to the most general possible, integro-differential isounit $\hat{I}(x, \dot{x}, \ddot{x}, \ldots)$.

One can therefore see from the very outset the richness and novelty of the isotopic theory. In fact, it can be classified into five main classes as occurring for isofields, isospaces, etc., and admits novel realizations and applications, e.g., in the construction of the symmetries of deformed line elements of metric spaces.

3.A. Isotopies and isodualities of universal enveloping associative algebras. Let ξ be a universal enveloping associative algebra [15] over a field F (of characteristic zero) with generic elements $A, B, C, \ldots$, trivial associative product AB and unit I. Their isotopes $\hat{\xi}$ were first introduced in [47] under the name of *isoassociative envlopes*. They coincide with ξ as vector spaces but are equipped with the isoproduct so as to admit $\hat{I}$ as the correct (right and left) unit

$$\hat{\xi} : A * B = ATB,\ T \text{ fixed}, I * A = A * I \equiv A\ \forall A\ \hat{\xi}, \hat{I} = T^{-1}. \quad \ldots (3.1)$$

Let $\xi = \xi(L)$ be the universal enveloping algebra of an N-dimensional Lie algebra L with ordered basis $\{X_k\}$, $k = 1, 2, \ldots, N$, $[\xi(L)]^- \approx L$ over F, and let the infinite-dimensional basis of $\xi(L)$ be given by the Poincare- Birkhoff-Witt theorem [15]. A fundamental result achieved by Santilli in the original proposal [47] (see also [59, Vol. II,p. 154-163] is the following

Theorem 3.1. *The cosets of* $\hat{I}$ *and the standard, isotopically mapped monomials*

$$\hat{I}, X_k\ \ X_i * X_j\, (i \leq j),\ \ X * X_j * X_k\ \ (i \leq j \leq k), \ldots \qquad \ldots (3.2)$$

form a basis of the universal enveloping isoassociative algebra $\hat{\xi}(L)$ *of a Lie algebra* L.

A first important consequence is that the isotopies of conventional exponentiation are given by the expression, called isoexponentiation, for $\hat{w} \in \hat{F}$,

$$e_{\xi}^{i\hat{w}*X} = \hat{I} + (i\hat{w}*X)/(1! + (i\hat{w}*X) * (i\hat{w}*X)/2! + \ldots = \hat{I}\,\{e^{iwTX}\} = \{e^{iXTw}\}\hat{I}. \quad \ldots (3.3)$$

As anticipated in Section1, the nontriviality is illustrated by the emergence of the nonlinear-nonlocal isotopic element T directly in the exponent of the transformations, thus ensuring the desired generalization.

The implications of Theorem 3.1 also emerge at the level of functional analysis because all structures defined via the conventional exponentiation must be suitably lifted into a form compatible with Theorem 3.1. As an example, Fourier transforms are structually dependent on the conventional exponentiation. As a result, they must be lifted under isotopies into the expression [23]

$$f(x) = (1/\pi) \int_{-\infty}^{+\infty} g(k) * e_{\xi}^{ikx}\, dk, g(k) = (1/2\pi) \int_{-\infty}^{+\infty} f(x) * e_{\xi}^{-ikx}\, dx, \quad \ldots (3.4)$$

with similar liftings for Laplace transforms, Dirac-delta distribution, etc., not reviewed here for brevity.

On physical grounds, Theorem 3.1 implies that the isotransform of a Gaussian in isofunctional analysis is given by [23]

$$f(x) = n * e_{\xi}^{-x/2a^2} = Ne^{-x^2T/2a^2} \Rightarrow g(k) = N * e\xi^{-k^2a^2/2} = N'e^{-k^2Ta^2/2}. \quad \text{... (3.5)}$$

As a result, the widths are of the type $\Delta x \approx aT^{-1/2}$, $\Delta k \approx a^{-1}T^{-1/2}$. It then follows that the isotopies imply the loss of the conventional uncertainties $\Delta x\, \Delta k \approx 1$ in favour of the local *isouncertainties* [61b].

$$\Delta x\, \Delta k \approx \hat{I}, \quad \text{... (3.6)}$$

which illustrate the nontriviality of the isotopy.

The *isodual isoenvelopes* $\hat{\xi}^d$ are characterized by the isodual basis $X_k^d = -X_k$ defined with respect to the isodual isounits $\hat{I}^d = -\hat{I}$ and isodual isotopic element $T^d = -T$ over the isodual isofields $\hat{F}^d$. The *isodual isoex-ponentiation* is the given by.

$$e_{\hat{\xi}^d}^{i^d \hat{w}^d x\, dX^d} = \hat{I}^d \{e^{iwTX}\} = -e_{\hat{\xi}}^{iwX} \quad \text{... (3.7)}$$

and plays an important role for the characterization of antiparticles as possessing negative-definite energy and moving backward in time(as necessary when using isodual isofields).

It is easy to see that Theorem 3.1 holds, as originally formulated [47], for envelopes now called of Class III, thus unifying isoenvelopes $\hat{\xi}$ and their isoduals $\hat{\xi}^d$. In fact, the theorem was conceived to unify with one single Lie algebra basis X_k nonisomorphic compact and noncompact algebras of the same dimension N (see the example of Section 3.E).

The isotopy $\xi \to \hat{\xi}$ is not a conventional map because the local coordinates x, the infinitesimal generators X_k and the parameters w_k are not changed by assumption, while the underlying unit and related associative product are changed. Also, in the operator realization the Lie and Lie-Santilli isotheory can be linked by nonunitary transformations $UU^{\tau} = \hat{I} \neq I$, for which.

$$I \to \hat{I} = UIU^{\tau}, AB - UABU^{\tau} = A' * B' = A'\, TB', T = (UU^{\tau})^{-1} \quad \text{... (3.8)}$$

where $A' = UAU^{\tau}$, $B' = UBU^{\tau}$. The lack of equivalence of the two theories is further illustrated by the inequivalence between conventional eigenvalue equations.

$$H \mid b > = E \mid b >, \ H = H^{\tau}, \ E \in \mathcal{R}(n, +, \times),$$

and their isotopic form in the same Hamiltonian

$$H * \mid \hat{b} > = HT \mid \hat{b} > \equiv \hat{E} \mid \hat{b} >, \ H = H^{\tau}, \ E' \in \mathcal{R}(n, +, \times),$$

with consequential *different eigenvalues for the same operator* $H, E' \neq E$ (see Section 3.E for an example). We therefore expect the weights of the Lie and Li-Santilli theories to be different.

3.B. Isotopies and isodualities of Lie algebras. A (finite-dimensional) isospace $\hat{L}$ over the isofield $\hat{F}$ of isoreal $\mathcal{R}(\hat{n}, +, *)$ or isocomplex numbers with isotopic element T and isounit $\hat{I} = T^{-1}$ is called a *Lie -Santilli algebra* over $\hat{F}$ (see [47], [49], [61], [62] for original studies and monographs [3], [24], [31], [74] with large number of quoted papers for independent studies), sametimes called isoalgebra (when no confusion with the isotopies of non- Lie algebras arises), when there is a composition $[A \hat{,} B]$ in $\hat{L}$, called *isocommutator*, which is isolinear (i.e. satisfies condition (2.40)) and such that for all $A, B, C \in \hat{L}$

$$[A \hat{,} B] = - [B \hat{,} A], \quad [A \hat{,} [B \hat{,} C]] + [B \hat{,} [C \hat{,} A]] + [C \hat{,} [A \hat{,} B]] = 0, \qquad \text{... (3.9a)}$$

$$[A * B \hat{,} C] = A * [B \hat{,} C] + [A \hat{,} C] * B. \qquad \text{... (3.9b)}$$

The isoalgebras are said to be *isoreal (isocomplex)* when $\hat{F} = \mathcal{R}(\hat{F} = \hat{C})$, and *isoabelian* when $[A \hat{,} B] \equiv 0 \ \forall \ A, B \in \hat{L}$. A subset $\hat{L}_0$ of $\hat{L}$ is said to be an *isosubalgebra* of $\hat{L}$ when $[\hat{L}_0 \hat{,} L_0] \subset \hat{L}_0$ and an isoideal when $[\hat{L} \hat{,} \hat{L}_0] \subset \hat{L}_0$. A maximal isoideal which verifies the property $[\hat{L} \hat{,} \hat{L}_0] = 0$ is called the *isocentre* of $\hat{L}$. For the isotopies of conventional notions, theorems and properties of Lie algebras, one may see monograph [74].

We recall the *isotopic generalizations of the celebrated Lie's First, Second and Third Theorems* introduced in the original proposal [47], but which we do not review here for brevity (see [49b], [61b], [74]. For instance, the isotopic second theorem reads

$$[X_i \hat{,} X_j] = X_i * x_j - X_j * X_i = X_i \, T(x, \ldots) X_j - X_j \, T(x, \ldots) X_i = \hat{C}_{ij}^k \, (x, \dot{x}, \dot{x}, ..) * X_k,$$

where the $\hat{C}$'s are called the *structure functions*, generally have an explicit dependence on the underlying isospace (see the example of Section 3.E.), and verify certain restrictions from the Isotopic Third Theorem.

Let L be an N-dimensional Lie algebra with conventional commutation rules and structure constants C_{ij}^k on a space $S(x, F)$ with local coordinates x over a field F, and let L be (homomorphic to) the antisymmetric algebra $[\xi(L)]^-$ attached to the associative envelope $\xi(L)$. Then $\hat{L}$ can be equivalently defined as (homomorphic to) the antisymmetric algebra $[\hat{\xi}(L)]^-$ attached to the isoassociative envelope $\hat{\xi}(L)$ ([47], [49], [74]). In this way, an infinite number of isoalgebras $\hat{L}$, depending on all possible isounits $\hat{I}$, can be constructed via the isotopies of one single Lie algebra L. It is easy to prove the following result:

Theorem 3.2 *The isotopies $L \to \hat{L}$ of an N-dimensional Lie algebra L preserve the original dimensionality.*

In fact, the basis e_k, $k = 1, 2, ..., N$ of a Lie algebra L is not changed under isotopy, except for renormalization factors denoted $\hat{e}_k$ Let the commutation rules of L be given by

$$[e_i, e_j] = C_{ij}^k e_k.$$

The isocomutation rules of the isotopes $\hat{L}$ are

$$[\hat{e}_i \,\hat{,}\, \hat{e}_i] = \hat{e}_i T \hat{e}_j - \hat{e}_j T \hat{e}_i = \tilde{C}_{ij}^k (x, \dot{x}, \dot{x}, ...) \hat{e}_k, \; C = \hat{C}T. \qquad ... (3.11)$$

One can then see in this way the necessity of lifting the structure <constants> into structure<functions> , as correctly predicted by the Isotopic Second Theorem.

The structure theory of the above isoalgebras is still unexplored to a considerable extent. In the following we shall show that the main lines of the conventional structure of Lie theory do indeed admit a consistent isotopic lifting. The begin, we here introduce the *general isolinear and isocomplex Lie-Santilli algebras* denoted $G\hat{L}(n, \hat{C})$ as the vecor isospaces of all $n \times n$ comlex matrices over $\hat{C}$. It is easy to see that they are closed under isocommutators as in the conventional case. The *isocentre* of $G\hat{L}(n, \hat{C})$ is then given by $\hat{a} * \hat{I}$, $\forall \hat{a} \in \hat{\mathfrak{R}}$. The subset of all complex $n \times n$ matrices with null trace is also closed under isocommutators. We shall call it the *special, complex, isolinear isoalgebra* and denote it with $S\hat{L}(n, \hat{C})$. The subset of all antisymmetric $n \times n$ real matrices $X, X^t = -X$, is also closed under isocommutators, it is called the *isoorthogonal algerba*, and it is denoted with $\hat{O}(n)$.

By proceeding along similar lines, we classify all classical, non-exceptional, Lie-Santilli algebras over an isofield of characteristic zero into the isotopes of the conventional forms, denoted with $\hat{A}_n, \hat{B}_n, \hat{C}_n$ and $\hat{D}_n$ each one admitting realizations of Classes I, II, III, IV and V (of which only Classes I, II and III are studied herein). In fact $\hat{A}_{n-1} = S\hat{L}(n, \hat{C})$; $\hat{B}_n = \hat{O}(2n + 1, \hat{C})$; $\hat{C}_n = S\hat{P}(n, \hat{C})$; and $\hat{D}_n = \hat{O}(2n, \hat{C})$.One can begin to see in this way the richness of the isotopic theory as compared to the conventional theory.

The notions of *homomorphism, automorphism and isomorphism* of two isoalgebras $\hat{L}$ and $\hat{L}'$, as well as of *simplicity and semisimplicity* are the conventional ones. Similarly, all properties of Lie algebras based on the addition, such as the *direct and semidirect sums, carry* over to the isotopic context unchanged (because of the preservation of the conventional additive unit 0).

An *isoderivation* $\hat{D}$ of an isoalgebra $\hat{L}$ is an isolinear mapping of $\hat{L}$ into itself satisfying the property

$$\hat{D}([A, B]) = [\hat{D}(A) \,\hat{,}\, B] + [A \,\hat{,}\, \hat{D}(B)] \; \forall A, B \in \hat{L}. \qquad ... (3.12)$$

If two maps $\hat{D}_1$ and $\hat{D}_2$ are isoderivations, then $\hat{a} * \hat{D}_1 + \hat{b} * \hat{D}_2$ is also an isoderivation, and the isocommutators of $\hat{D}_1$ and $\hat{D}_2$ is also an isoderivation. Thus, the set of all isoderivations forms a Lie-Santilli algebra as in the conventional case.

The isolinear map $a\hat{d}(\hat{L})$ of $\hat{L}$ into itself defined by

$$a\hat{d}\, A(B) = [A \hat{,} B], \ \forall\, A, B \in \hat{L}, \qquad \text{... (3.13)}$$

is called the *isoadjoint map*. It is an isoderivation, as one can prove via the iso-Jacobi identity. The set of all $a\hat{d}(A)$ is therefore an isolinear isoalgebra, called *isoadjoint algebra* and denoted $\hat{L}_a$. It also results to be an isoideal of the algebra of all isoderivations as in the conventional case.

Let $\hat{L}^{(0)} = \hat{L}$. Then $\hat{L}^{(1)} = [\hat{L}^{(0)} \hat{,} \hat{L}^{(0)}]$, $\hat{L}^{(2)} = [\hat{L}^{(1)}, \hat{L}^{(1)}]$, etc., are also isoideals of $\hat{L}.\hat{L}$ is then called *isosolvable* if, for some positive integer n, $\hat{L}^{(n)} \equiv 0$. Consider also the sequence.

$$L_{(0)} = L, \hat{L}_{(1)} = [\hat{L}_{(0)} \hat{,} \hat{L}], \ \hat{L}_{(2)} = [\hat{L}_{(1)} \hat{,} \hat{L}], \ \text{etc.},$$

Then $\hat{L}$ is said to be *isonilpotent* if, for some positive integer n, $\hat{L}(n) \equiv 0$. One can then see that, as in the conventional case, an isonilpotent algebra is also isosolvable, but the converse is not necessarily true.

Let the isotrace of a matrix be given by the element of the isofield [61]

$$T\hat{r}A = (Tr\, A)\hat{I} \in \hat{F}, \qquad \text{... (3.14)}$$

where $Tr\, A$ is the conventional trace. Then

$$T\hat{r}\,(A * B) = (T\hat{r}\,A) * (T\hat{r}\,B), \ \ T\hat{r}\,(BAB^{-1}) = T\hat{r}\,A.$$

Thus, the $T\hat{r}\,A$ preserves the axioms of $Tr\ A$, by therefore being a correct isotopy. Then the isoscalar product

$$(A \hat{,} B) = T\hat{r}[(A\hat{d}X) * (a\hat{d}\, B)], \qquad \text{... (3.15)}$$

is here called the *isokilling form*. It is easy to see that $A \hat{,} B)$is symmetric, bilinear, and verifies the property $A\hat{d}X(Y) \hat{,} Z) + (Y \hat{,} Ad\, X(z)) = 0$, thus being a correct, axiom-preserving isotopy of the conventional Killing form.

Let e_k, $k = 1, 2, \ldots N$, N, be the basis of L with oneto-one invertible map $e_k \rightarrow \hat{e}_k$ to the basis of $\hat{L}$. Generic elements in $-\hat{L}$ can then be written in terms of local coordinates $x, y, z, A = x^i \hat{e}_i$ and $B = y^j \hat{e}_j$, and

$$C = z^k\, \hat{e}_k = [A \hat{,} B] = x^i\, y^j\, [\hat{e}_i \hat{,} \hat{e}_j] = x^i\, x^j\, \tilde{C}^k_{ij}\, \hat{e}_k.$$

Thus,

$$[A\hat{d}\, A(B)]^k = [A \hat{,} B]^k = \tilde{C}^k_{ij}\, x^i\, x^j. \qquad \text{... (3.16)}$$

We now introduce the *isocartan tensor* $\tilde{g}_{ij}$ of an isoalgebra $\hat{L}$ via the definition $(A \hat{,} B)(B) = \tilde{g}_{ij} x^i y^j$ yielding

$$\tilde{g}_{ij}(x, \dot{x}, \ddot{x}, \ldots) = \tilde{C}^k_{ip} C_{jk} p. \qquad \ldots (3.17)$$

Note that the isocartan tensor has the general dependence of the isometric tensor of Section 2.*C*, thus confirming the inner consistency among the various branches of the isotopic theory. In particular, the isocartan tensor is generally *nonlinear, nonlocal and noncanonical* in all variables $x, \dot{x}, \ddot{x}, \ldots$. This clarifies that isotopic generalization of the Riemannian spaces studied in the companion paer [60] $\hat{\mathfrak{R}}(\hat{x}, \hat{g}, \hat{\mathfrak{R}})$, $\hat{g} = \hat{g}(s, x, \dot{x}, \ddot{x}, \ldots)$, has its origin in the very structure of the Lie-isotopic theory.

The isocartan tensor also clarifies another fundamental point of Section 1, that the isotopies naturally lead to an arbitrary dependence in the velocities and accelerations, exactly as needed for realistic treatment of the problems identified in Section 1, and that their restriction to the nonlinear dependence on the coordinates x only, as generally needed for the exterior(e.g., gravitational)problem, would be manifestly un-necessary.

The isotopies of the remaining aspects of the structure theory of Lie algebras can be completed by the interested reader. Here we limit ourselves to recall that when the isocartan form is positive- (or negative-) definite, $\hat{L}$ is compact, otherwise it is noncompact. Then it is easy to prove the following.

Theorem 3.3 *The Class III liftings $\hat{L}$ of a compact (noncompact) Lie algebra L are not necessarily compact (noncompact).*

The identification of the remaining properties which are not preserved under liftings of Class III is an instructive task for the interested reader. For instance, if the original structure is irreducible, its isotopic image is not necessarily so even for Class I, trivially, because the isounit itself can be reducible, thus yielding a reducible isotopic structure.

Let $\hat{L}$ be an isoalgebra with generators X_k and isounit $\hat{I} = T^{-1} > 0$. From Equations (3.7) we then see that the *isodual Lie-Santilli algebras* $\hat{L}^d$ of $\hat{L}^d$ is characterized by the isocommutators

$$[X_i \hat{,} X^d_{i]} = -[X_i \hat{,} X_j] = \tilde{C}^{k\,(d)}_{ij} X^d_k, \;\; \tilde{C}^{k(d)}_{ij} = -\tilde{C}^k_{ij}. \qquad \ldots (318)$$

$\hat{L}$ and $\hat{L}^d$ are then (anti) isomorphic. Note that the isoalgebras of Class III contain all class I isoalgebras $\hat{L}$ and all their isoduals $\hat{L}^d$. The above remarks therefore show that the Lie-Santilli theory can be naturally formulated for Class III, as implicity done in the original proposal [47]. The formulation of the same theory for Class IV or V is however considerably involved on technical grounds thus requiring specific studies.

The notion of isoduality applies also to conventional Lie algebras L, by permitting the identification of the *isodual Lie algebras* L^d via the rule [52], [53]

$$[X_i, X_j]^d = X^d{}_i\, I^d X^d{}_j - X^d{}_j\, I^d X^d_i = -\,[X_i, X_j] = C_{ij}^{k(d)}\, X_k^d,\ \ C_{ij}^{k(d)} = -\,C_{ij}^k.$$

Note the necessity of the isotopies for the very construction of the isodual of conventional Lie algebras. In fact, they require the nontrivial lift of the unit $I \Rightarrow I^d = (-I)$, with consequential necessary generalization of the Lie product $AB - BA$ into the isotopic form $ATB - BTA$.

The following property is mathematically trivial, yet carries important physical applications.

Theorem 3.4 *All infinitely possible, Class I isotopes $\hat{L}$ of a (finite- dimensional) Lie algebra L are locally isomorphic to L and all infinitely possible, Class II isodual isotopes $\hat{L}^d$ are anti- isomorphic to L.*

As outlined in Section $2F$, the classical realization of the formulation of this section is provided by functions $X_i, X_j, \ldots$ on the isotangent bundle $T * \hat{E}(r, \hat{\delta}, \hat{\mathcal{R}}), \hat{\delta} = T\delta$, with local chart $a = \{p_k, r^k\}$ and isoalgebra

$$[X_i \hat{,} X_j] = \frac{\partial X_i}{\partial x^m} T^n_m(x, p, \ldots) \frac{\partial X_j}{\partial p_n} - \frac{\partial X_j}{\partial x^m} T^n_m(x, p, \ldots) \frac{\partial X_i}{\partial p_n} = \hat{C}^k_{ij}\, T^m_k X_m \ldots \quad (3.19)$$

As outlined in Sect. 2.G, the operator realization is given by operators X_k on an isohilbert space with a given isounit $\hat{I} = T^{-1}$ and isoalgebra

$$[X_i \hat{,} X_j] = X_i\, T\, X_j - X_j\, T\, X_i = \hat{C}^k_{ij}\, TX_k. \qquad \ldots (3.20)$$

The unique and unambiguous map interconnection realizations (3.19) and (3.20) is the isoquantization of Sect. 2G.

3.C. Isotopies and isodualities of Lie groups.

A *right Lie-Santilli group* $\hat{G}$ (See [47], [49], [61], [62] for original studies and monographs [3], [24], [31], [74] with large number of quoted papers for independent studies) on an isospace $\hat{S}(x, \hat{F})$ over an isofield $\hat{F}, \hat{I} = T^{-i}$ (of isoreal $\hat{\mathcal{R}}$ or isocomplex numbers $\hat{C}$), also called *isotransformation group of isogroup,* is a group which maps each element $x \in \hat{S}(x, \hat{F})$ into a new element $x' \in \hat{S}(x, \hat{F})$ via the isotransformations $x' = \hat{U} * x = \hat{U}Tx$, T fixed, such that : (1) The map $(U, x) \rightarrow \hat{U} * x$ of $\hat{G} \times \hat{S}(x, \hat{F})$ onto $\hat{S}(x, \hat{F})$ is isodifferentiable (2) $\hat{I} * \hat{U} = \hat{U} * \hat{I} = \hat{U}\ \ \forall\ \ \hat{U} \in \hat{G}$; and (3) $\hat{U}_1 * (\hat{U}_2 * x) = (\hat{U}_1 * \hat{U}_2) * x,\ \ \forall\ x \in \hat{S}(x, \hat{F})$ and $\hat{U}_1, \hat{U}_2 \in \hat{G}$. *A left isotransformation group is defined accordingly.*

The notions of *connected or simply connected transformation groups* carry over to the isogroups in their entirety. We consider hereon the connected isotransformation groups. Right or left isogropus are characterized by the following laws [47].

$$\hat{U}(0) = \hat{I},\ \hat{U}(\hat{w}) * \hat{U}(\hat{w}') = \hat{U}(\hat{w}') * \hat{U}(\hat{w}) = \hat{U}(\hat{w} + \hat{w}'),\ \hat{U}(\hat{w}) * \hat{U}(-\hat{w}) = \hat{I},\ \hat{w} \in \hat{F}. \quad \text{... (3.21)}$$

Their most direct realization of the isotransformation groups is that via isoexponentiation (3.3),

$$\hat{U}(\hat{w}) = \prod_k e_\xi^{i\hat{w}_k * X_k} = \prod_k e_\xi^{i} X_k * \hat{w}_k = \hat{I}\{\prod_k e^{iw_k TX_k}\} = \prod_k e^{iX_k Tw_k}\}\hat{I} \quad \text{... (3.22)}$$

where the X's and w's are the infinitesimal generator and parameters, respectively, of the original algebra L. Equations (3.22) hold for some open neighbourhood of N of the isoorigin of $\hat{L}$ and, in this way, characterize some open neighbourhood of the isounit of $\hat{G}$. Then the isotransformations can be reduced to an ordinary transform for computational convenience,

$$x' = \hat{U} * x = \{\prod_k e_\xi^{iX_k * w_k}\} * x = \{\prod_k e^{iX_k Tw_k}\}x, \quad \text{... (3.23)}$$

with the understanding that, on rigorous mathematical grounds, only the isotransform is correct.

Still another important result obtained in [47] is the proof that conventional group composition laws admit a consistent isotopic lifting, resulting in the following isotopy of the Baker-Campbell-Hausdorff Theorem

$$\{e_\xi^{X}\} * (e_\xi^{X}\} = e_\xi^{X_3},\ X_3 = X_1 + X_2 + [X_1 \hat{,} X_2]/2 + [(X_1 - X_2) \hat{,} [X_1 \hat{,} X_2]]/12 + \ldots \quad \text{... (3.24)}$$

Note the crucial appearance of the isotopic element $T(x, \dot{x}, \ddot{x}, \ldots)$ in the exponent of the isogroup. This ensures a structural generalization of Lie's theory of the desired nonlinear, nonlocal and noncanonical form. For details see [49] and [74].

The structure theory of isogroups is also vastly unexplored at this writing. In the following we shall point out that the conventional structure theory of Lie groups does indeed admit a consistent isotopic lifting. The isotopies of the notions of weak and strong continuity of [22] are a necessary pre-requisite. Let $\hat{L}$ be a (finite-dimensional) Lie-Santilli algebra with (ordered) basis $\{X_k\}$, $k = 1, 2, \ldots, N$. For a sufficiently small neighborhood N of the isoorigin of $\hat{L}$, a generic element of $\hat{G}$ can be written

$$\hat{U}(w) = \prod^{*} k = 1, 2, \ldots, N\, e_\xi^{iX_k w_k}, \quad \text{... (3.25)}$$

which characterizes some open neighborhood M of the isounit $\hat{I}$ of $\hat{G}$ The map

$$\hat{\Phi}_{\hat{U}_1}(\hat{U}_2) = \hat{U}_1 * \hat{U}_2 * \hat{U}_1^{-\hat{I}}, \qquad \text{... (3.26)}$$

for a fixed $\hat{U}_1 \in \hat{G}$, characterizes an *inner isoautomorphism* of $\hat{G}$ onto $\hat{G}$. The corresponding isoautomorphism of the algebra $\hat{L}$ can be readily computed by considering the above expression in the neighborhood of the isounit $\hat{I}$. In fact, we have

$$\hat{U}'_2 = \hat{U}_1 * \hat{U}_2 * \hat{U}_1^{-\hat{I}} \approx \hat{U}_2 + w_1 w_2 [X_2 \hat{,} X_1] + O^{(2)}. \qquad \text{... (3.27)}$$

The reduction of the isogroups to isoalgebras requires the knowledge of isodifferentials $\hat{d}w = Tdw$ and isoderivatives $\hat{d}/\hat{d}w = \hat{I}dw$, under which we have the following expression in one dimension :

$$i^{-1} \frac{\hat{d}}{\hat{d}w} \hat{U} \mid_{w=0} = X * e_{\xi}^{\ iwX} \mid_{w=0} = X. \qquad \text{... (3.28)}$$

where we have used the isodifferential $\hat{d}w_k = T_k^{\ i}\, dw_i$ and related isoderivative (Sect. 2.C).

Thus, to every inner isoautomorphism of $\hat{G}$, there corresponds an inner isoautomorphism of $\hat{L}$ which can be expressed in the form :

$$(\hat{L})^j_i = \tilde{C}^j_{ki}\, w^k. \qquad \text{... (3.29)}$$

The isogroup $\hat{G}_a$ of all inner isoautomorphism of $\hat{G}$ is called the isoadjoint group. It is possible to prove that the Lie-Santilli algebra of $\hat{G}_a$ is the isoadjoint algebra $\hat{L}_a$ of $\hat{L}$. This establishes that the connections between algebras and groups carry over in their entirety under isotopies.

We mentioned before that the direct of isoalgebras is the conventional operation because the addition is not lifted under isotopies (otherwise there will be the loss of distributivity, see [59]. The corresponding operation for groups is the semidirect product which, as such, demands care in its formulation.

Let $\hat{G}$ be an isogroup and $\hat{G}_a$ the group of all its inner isoautomorphisms. Let $\hat{G}^0_a$ be a subgroup of $\hat{G}_a$, and let $\hat{\Lambda}(\hat{G})$ be the image of $\hat{g} \in \hat{G}$ under $\hat{G}^0_a$. The semidirect isoproduct $\hat{G} \hat{\times} \hat{G}^0_a$ of $\hat{G}$ and $\hat{G}^0_a$ and is the isogroup of all ordered pairs

$$(\hat{g}, \hat{\Lambda}) * (g', \hat{\Lambda}) = (\hat{g} * \hat{\Lambda}(\hat{g}'), \hat{\Lambda} * \hat{\Lambda}'), \qquad \text{... (3.30)}$$

with total isounit given by $(\hat{I}, \hat{I}_{\hat{\Lambda}})$ and invese $\hat{g}, \hat{\Lambda}s)^{-\hat{1}}$ $= (\hat{\Lambda}^{-\hat{1}} = (\hat{\Lambda}^{-\hat{1}} (\hat{g}^{-\hat{1}}), \hat{\Lambda}^{-1}$. The above notion plays an important role in the isotopies of the inhomogeneous space-time symmetries outlined later on.

Let $\hat{G}_1$ and $\hat{G}_2$ be two isogroups with respective isounits $\hat{I}_1$ and $\hat{I}_2$. The *direct isoproduct* $\hat{G}_1 \hat{\otimes} \hat{G}_2$ of $\hat{G}_1$ and $\hat{G}_1$ is the isogroup of all ordered pairs $(\hat{g}_1, \hat{g}_2), \hat{g}_1 \in \hat{G}_1, \hat{g}_2 \in \hat{G}_2$, with isomultiplication

$$(\hat{g}_1, \hat{g}_2) * (\hat{g}'_1, \hat{g}'_2) = (\hat{g}_1 * \hat{g}'_1, \hat{g}_2 * \hat{g}'_2), \qquad \ldots (3.31)$$

total isounit $(\hat{I}_1, \hat{I}_2)$ and inverse $(\hat{g}_1^{-\hat{I}}, \hat{g}_2^{-\hat{I}})$, The isotopies of the remaining aspects of the structure theory of Lie groups can then be investigated by the interested reader.

Let $\hat{G}$ be an N-dimensional isotransformation group of Class I with infinitesimal generators X_k, $k = 1, 2, .., N$. The isodual Lie-Santilli group $\hat{G}^d$ of $\hat{G}$ ([52], [53]) is the N-dimensional isogroup with generators $X_k^d = -X_k$ constructed with respect to the isodual isounit $\hat{I}^d = -\hat{I}$ over the isodual isofield $\hat{F}^d$. By recalling that $w \in F \Rightarrow w^d \in F^d, w^d = -w$, a generic element of $\hat{G}^d$ in a suitable neighborhood of $\hat{I}^d$ is therefore given by

$$\hat{U}^d(\hat{w}^d) = e_\xi d^{i^d \hat{w}^d * X^d} = -e_\xi^{i\hat{w} * X} = -\hat{U}(\hat{w}). \qquad \ldots (3.32)$$

The above antiautomorphic conjugation can also be defined for conventional Lie group, yielding the *isodual Lie group* G^d of G with generic elements $U^d(w^d) = e_\xi d^{iw^d X} = -e_\xi^{iwX}$.

The symmetries significant for this paer are the following ones: the conventional form G, its isodual G^d, the isotopic form $\hat{G}$ and the isodual isotopic form $\hat{G}^d$. These different forms are useful for the respective characterization of particles and antiparticles in vacuum (exterior problem) or within physical media (interior problem).

It is hoped that the reader can see from the above elements that the entire conventional Lie's theory does indeed admit a consistent and nontrivial lifting into the covering Lie-Santilli formulation. Particularly important are the isotopies of the conventional representation theory, known as the *isorepresentation theory*, which naturaly yields the most general known, nonlinear, nonlocal and noncanonial representations of Lie groups. Studies along these latter lines were initiated by Santilli with the isorepresentations of $S\hat{U}(2)$ and of $S\hat{U}(3)$ [61], by Klimyk and Santilli Klimyk [27], and others.

As reviewed in Sect. 2.F, a classical realization of the formulation of this section is given by the isotangent bundle $T * \hat{E}(r, \hat{\delta}, \hat{R}, \hat{\delta} = T\delta$, with local chart and isounit

$$a = \{r^k, p_k\}, \mu = 1, 2, 3, 4, 5, 6, k = 1, 2, 3, \quad \hat{I}_2 = \text{diag.}\,(\hat{I}, \hat{T}). \; (3.33)$$

the Hamilton-Santilli equations

$$\hat{d}a^\mu / \hat{d}t = w^{\mu a}\, T^\nu_{2a} \frac{\partial H}{\partial a^\nu}, \qquad \ldots (3.34)$$

where $w^{\mu\alpha}$ is the familiar canonical Lie tensor and their exponen tiated form

$$a(t) = (e^{tw^{\mu\alpha}T^{\nu}_{2\alpha}(\partial H/\partial a^{\mu})^{\partial/\partial a^{\alpha}}}\}\, a(0), \qquad \text{... (3.35)}$$

where we have ignored the factorization of the isounit in the isoexponent for simplicity.

As recalled in Sect. 2*G*, an operator realization of the Lie-Santilli isogroups is given by *isounitary transformations* $x' = \hat{U} * x$ on an isohibert space $\mathcal{H}$ with

$$\hat{U} * \hat{U}^{\tau} = \hat{U}^{\tau} * \hat{U} = \hat{I}, \qquad \text{... (3.36)}$$

with realization in terms of an *isohermitean operator H*

$$\hat{U} = \hat{e}^{iHt} = \{e^{iHTt}\}\hat{I}. \qquad \text{... (3.37)}$$

The above classical and operator realizations are also interconnected in a unique and unambiguous way by the isoquantzation (Sect. 2.*G*).

3.D. Santilli's fundamental theorem on isosymmetries. We are now equipped to review without proof the following important result formulated in [52] and then studied in details in [61] and [62]

Theorem 3.5. *Let G be an N-dimensional Lie group of isometries of an m-dimensional metric or pseudo-metric space S(x, g, F) over a field F*

$$G : x' = A(w)\, x,\ \ (x' - y')\tau\, A\, \tau\, gA(x - y) \equiv (x - y)\tau\, g(x - y), A\tau\, g\, A$$
$$AgA\tau = g, \qquad \text{... (3.38)}$$

Then the infinitely possible isotopies $\hat{G}$ of G of Class III characterized by the same generators and parameters of G and new isounits $\hat{I}$ (isotopic elements T), automatically leave invariant the isocomposition on the isospaces $\hat{S}(x, \hat{g}, \hat{F}), \hat{g} = Tg\,\hat{I} = T^{-1}$,

$$\hat{G} : x' = \hat{A}(w) * x, (x' - y')\hat{U}* \hat{A}\hat{U}\hat{g}\,\hat{A} * (x - y) = (x - y)\hat{U}\hat{g}\,(x - y), \hat{A}\hat{U}\hat{g}\,\hat{A}$$
$$= \hat{A}\,\hat{g}\,\hat{A}\,\hat{U} = \hat{I}\,\hat{g}\,\hat{I}, \qquad \text{... (3.39)}$$

The "direct universal" of the resulting isosymmetries for all infinitely possible isotopies $g \to \hat{g}$ is then evident owing to the completely unrestricted functional dependence of the isostopic element T in the isometric $\hat{g} = Tg$. One should also note the insufficiency of the so-called trivial isotopy

$$X_k - X'_k = X_k\,\hat{I}, \qquad \text{... (3.40)}$$

for the achievement of the desired form-invariance. In fact, inder the above mapping the isoexponentiation becomes

$$e_{\xi}^{iX'_k * w_k} = \{e^{iX'_k Tw_k}\}\hat{I} = \{e^{iX_k w k}\}\hat{I}, \qquad \text{... (3.41)}$$

namely, we have the disappearance precisely of the isotopic element T in the exponent which provides the invariance of the isoseparation.

3.E. Isotopies and isodualities of the rotational symmetry. We now illustrate the Lie- Santilli isotheory with the first mathematically and physically significant case, *the isotopies of the rotational symmetry, also called isorotational symmetry*. They were first achieved in [53] and then studied in details in [61], including the isotopies of SU(2), their isorepresentations, the iso-Clebsh-Gordon coefficients, etc.

Consider the lifting of the perfect sphere in Euclidean space $E(r, \delta, \mathcal{R})$ with local coordinates $r = (x, y, z)$, and metric $\delta = diag.\ (1, 1, 1)$ over the reals $\mathcal{R}$.

$$r^2 = r^t \delta r = xx + yy + zz, \qquad \text{... (3.42)}$$

into the most general possible ellipsoid of Class III on isospace $\hat{E}^{III}(r, \hat{\delta}, \hat{R})$, $\hat{\delta} = T\delta$, $T = \text{diag.}\ (g_{11} \cdot g_{22}, g_{33})$, $\hat{I} = T^{-1}$.

$$r^{\hat{2}} = r^t\, \hat{\delta}\, r = xg_{11}\, y + y\, g_{22}\, y + z\, g_{33} z,\ \delta^{\tau} = \hat{\delta},\ \hat{}g_{kk} = g_{kk}(t, r, \dot{r}, \ddot{r}, ...) \neq 0, \qquad \text{... (3.43)}$$

The invariance of the original separation r^2 is the conventional rotational symmery $O(3)$. The isotopic techniques then permit the construction, in the needed explicit and finite form, of the isosymmetries $\hat{O}(3)$ of all infinitely possible generalized invariants $r^{\hat{2}}$ via the following steps: (1) Identification of the basics isotopic element T in the lifting $\delta \to \hat{\delta} = T\delta$, which, in this particular case, is given by the new metric $\hat{\delta}$ itself, $T \equiv \hat{\delta}$ and identification of the fundamental unit of the theory. $\hat{I} = T^{-1}$; (2). Consequential lifting of the basic field $\hat{\mathcal{R}}(n, +, \times) \Rightarrow \hat{\mathcal{R}}(\hat{n}, +, *)$; (3) Identification of the isospace in which the generalized metric $\hat{\delta}$ is defined, which is given by the three-dimensional isoeuclidean spaces $\hat{E}(r, \hat{\delta}, \hat{\mathcal{R}})$, $\hat{\delta} = T\delta$, $\hat{I} = T^{-1}$; (4). Construction of the $\hat{O}(3)$ symmetry via the use of the original parameters of $O(3)$ the Euler's angles θ_k, $k = 1, 2, 3)$, the original generators (the angular momentum components $M_k = \in k_{ij}\, r^i p_j)$ in their fundamental (adjoint) representation, and the new metric δ; and (5) Classification, interpretation and application of the results.

The explicit construction of $\hat{O}(3)$ is straightforward. According to the Lie-Santilli theory, the connected component $S\hat{O}(3)$ of $\hat{O}(3)$ is given by [53]

$$S\hat{O}(3) : r' = \hat{R}(\theta) * r,\ \hat{R}(\theta) * r,\ \hat{R}(\theta) = \prod^{*}_{k=1,2\,3} e_{\xi}^{iM_k \theta_k} = (\prod_{k=1,2,3} e^{iM_k T\theta_k}\}\hat{I}, \qquad \text{... (3.44)}$$

while the discrete component is given by the *isoinversions* [loc. cit.] $r' = \hat{\pi} * r = \pi r = -r$, where π is the conventional inversion.

Under the assumed conditions on the isotopic element T, the convergence of isoexponentiations is ensured by the original convergence, thus permitting the explicit construction of the isorotations, with example around the third axis [53]

$$x' = x \cos [\theta_3(g_{11}g_{22})^{1/2}] + y\, g_{22}(g_{11} g_{22})^{-1/2} \sin [\theta_3(g_{11} g_{22})^{1/2}],$$
$$y' = -x\, g_{11} (g_{11} g_{22})^{-1/2} \sin [\theta_3(g_{11} g_{22})^{1/2}] + y \cos [\theta_3[g_{11} g_{22})^{1/2}],$$
$$z' = z. \qquad \text{... (3.45)}$$

(see [61b] for general isorotations). One should note that the argument of the trigonometric functions as derived via the above isoexponentiation coincides with the isoangle of the isotrigonometry in $\hat{E}(r, \delta, \hat{\mathcal{R}})$ (see paper [60] thus confirming the remarkable compatibility and interconnections of the various branches of the isotopic theory.

The computation of the isoalgebras $\hat{o}(3)$ of $\hat{O}(3)$ is then straightforward [loc. cit.]. In fact, when M_k are assumed to be in their regular representation we have [53]

$$\hat{o}(3)\text{: } [M_i \hat{,} M_j] = M_i\, T\, M_j - M_j\, T\, M_i = \hat{C}^k_{ij} * M_k,\ \hat{C}^k_{ij} = \in ijk\, g^{-1}_{kk}\, \hat{I}. \qquad \text{... (3.46)}$$

The above isoalgebra illustrates the explicit dependence of the structure *functions*. The proof of the isomorphism $\hat{o}(3) \approx o(3)$ as done [loc. cit.] via a suitable reformulation of the basis under which the structure functions recover the value $\hat{\in}_{ijk} = \in_{ijk} \hat{I}$.

The isocenter of $s\hat{o}(3)$ is characterized by the *isocasimir invariants*

$$C^{(0)} = \hat{I},\ C^{(2)} = M^{\hat{2}} = M * M = \Sigma_{k = 1, 2, 3}\, M_k\, T\, M_k. \qquad \text{... (3.47)}$$

In hadronic mechanics the realization is the following. The linear momentum operator has the isotopic form for a diagonal isounit

$$p_k * \mid \hat{\psi} > = -i\, \hat{\nabla}_k \mid \hat{\psi} > = -i\hat{I}^i_k\, \nabla_i \mid \hat{\psi} > .$$

The fundamental isocommutation rules are then given by

$$[r^i \hat{,} p_j] = i\delta^i_j = i\hat{I}\delta^i_j,\ [r^i \hat{,} r^j] = [p_i \hat{,} p_j] = 0.$$

However, in their contravariant form the coordinates are given $r_k = \hat{\delta}_{ki}\, r^i$. As a result $\hat{\nabla}_i\, r_j = \delta_{ij}$ (where the delta is the *conventional* Kronecker delta). In this case the fundamental isocommutation rules are given by

$$[r_i \hat{,} p_j] = i\delta^i_j = i\hat{I}\delta^i_j,\ \ [r_i \hat{,} r_j] = [p_i \hat{,} p_j] = 0,$$

namely, their eigenvalues *coincide* with the quantum ones. The operator isoalgebra $\hat{o}(3)$ with generators $M_k = \in_{kij}\, r_i p_j$

$$\hat{o}(3) : [M_i \hat{,} M_j] = M_i\, T\, M_j - M_j\, T\, M_i = i \in^k_{ij} * M_k,\ \hat{\in}^k_{ij} = \in_{ijk} \hat{I},$$

namely the *product of the algebra is generalized, but the structure consants are the conventional ones* (see [61] for details). The above results illustrates again the abstract identity of quantum and hadronic mechanics.

Note the nonlinear-nonlocal-noncanonical character of isotransformations (3.45) owing to the unrestricted functional dependence of the diagonal elements g_{kk}. Note also the extreme simplicity of the final results. In fact, the explicit symmetry transformations of separation (3.43) are provided by just plotting the given g_{kk} values into transformations (3.45) without any need of any additional computation. Note finally that the above invariance includes as particular case the general isosymmetry $\hat{O}(3)$ of (the space-component of) gravitation which, since it is locally Euclidean,remains isomorphic to $O(3)$.

As an example, the symmetry of the space-component of the Schwartzschild line element is given by plotting the following values

$$g_{11} = (1 - M/r)^{-1}, g_{22} = r^2, g_{33} = r^2 \sin^2 \theta, \qquad \text{... (3.48)}$$

(see next section for the full (3+1)- dimensional case).

Despite this simplicity, the implications of the above results are nontrivial. On physical grounds, the isounit $\hat{I} > 0$ permits a direct representation of the nonspherical shapes, as well as all their infinitely possible deformations. By recalling that 0(3) is a *theory of rigid bodies*, $\hat{O}(3)$ results to be a *theory of deformable bodies* (53) with fundamentally novel physical applications in the theory of elasticity, nuclear, particle physics, crystallography, and other fields [61], [62].

On mathematical grounds, we have equally intriguing novel insights. To see them, one must first understand the background isogeometry $\hat{E}^{III}(r, \hat{\delta}, \hat{\mathcal{R}})$ which unifies all possible conics in $E(r, \delta, \mathcal{R})$ as mentioned earlier. To be explicit in this important point, the geometric differences between (oblate or prolate) ellipsoids and (elliptic or hyperbolic) paraboloids have mathematical sense when projected in our Euclidean space $E(r, \delta, \mathcal{R})$. However all these surfaces are geometrically unified with the perfect isosphere in $\hat{E}(r, \hat{\delta}, \hat{\mathcal{R}})$.

These geometric occurrences permits the unification of **O**(3) and $O(2.1)$, as well as of all their infinitely possible isotopes. In fact, the classification of all possible isosymmetries $\hat{O}(3)$, achieved in the original derivation [53], includes :

(1) The compact $O(3)$ symmetry evidently for $\hat{\delta} = \delta = \text{diag.}\ (1,1,1)$;

(2) The noncompact $O(2.1)$ symmetry evidently for $\hat{\delta} = \ \text{diag.}\ (1,1,-1\}$

(3) The isodual $O^d(3)$ of $O(3)$ holding for $\hat{\delta} = \text{diag.}\ (-1,-1,-1)$;

(4) The isodual $O^d(2.1)$ of $O(2.1)$ holding for $\hat{\delta} = \text{diag.}\ (-1,-1,1)$;

(5) The infinite family of compact isotopes $\hat{O}(3) \approx O(3)$ with $I > 0$ for

$$\hat{\delta} = \text{diag.}\ (b_1^2, b_2^2, b_3^2), \quad b_k > 0;$$

(6) The infinite family of noncompact isotopes $\hat{O}(2.1) \approx O(2.1)$ for

$$\delta = \text{diag.}\ (b_1^2, b_2^2, - b_3^2);$$

(7) The infinite family of compact isodual isotopes $\hat{O}^d(3) \approx O^d\ (3)$ for

$$\hat{\delta} = \text{diag.}\ (- b_1^2, - b_2^2, - b_3^2);$$

(8) The infinite family of isodual isotopes $\hat{O}^d\ (2.1) \approx O^d\ (2.1)$ for

$$\hat{\delta} = \text{diag.}\ (- b_1^2, - b_2^2, b_3^2).$$

Even greater differentiations between the Lie and Lie-Santilli theories occur in their representations because of the change in the eigenvalue equations due to the nonunitarity of the map indicated in Sect. 1, from the familiar form $H\psi = E^0\psi$, isotopic form $H{*}\hat{\psi} = \hat{E} * \hat{\psi} \equiv E\hat{\psi}, E^0 \neq E$), thus implying generalized weights, Cartan tensors and other structures studied earlier

The first differences emerge in the spectrum of eigenvalues of $\hat{O}(2)$ and $O(2)$. in fact, the $o(2)$ albegra on a conventional Hilbert space solely admits the spectrum M = 0, 1, 2, 3 (as a necessary condition of unitarity). For the covering $\hat{O}(2)$ isoalgebra on an isohilbert space with isotopic element $T = \text{Diag.}\ (g_{11}, g_{22})$, the spectrum is instead given $\hat{M} = g_{11}^{-1/2}\, g_{22}^{-1/2}\, M$ and, as such, it can acquire *continuous* values in a way fully consistent with the condition, this time, of isounitarity. For the general $\hat{O}(3)$ case see also the detailed studied of ref. [61].

Similarly, the unitary irreducible representations of su(2) are characterize the familiar eigenvalues

$$J_3\hat{\psi} = M\psi,\ J^2\psi = J(J+1)\psi,\ M = J, J-1, \ldots, -J,\ J = 0, \tfrac{1}{2}, 1, \ldots \ldots \qquad (3.49)$$

Three classes of irreducible isorepresentation of $s\hat{u}(2)$ were identified in [63] which, for the adjoint case, are given by the following generalizations of Pauli's matrices:

(1) *Regular isopauli matrices*

$$\hat{\sigma}_1 = \Delta^{-1/2}\begin{pmatrix} 0 & g_{11} \\ g_{22} & 0 \end{pmatrix} \hat{\sigma}_2 = \Delta^{-1/2}\begin{pmatrix} 0 & -ig_{11} \\ ig_{22} & 0 \end{pmatrix}, \hat{\sigma}_3 = \Delta^{-1/2}\begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} \qquad \ldots (3.50a)$$

$$T = \text{diag.}\ (g_{11}, g_{22}),\ \ \Delta = \det T = g_{11}\, g_{22} > 0,\ \ [\hat{\sigma}_i, \hat{\sigma}_j]^{\xi} = i2\Delta^{1/2} \in_{ijk} \hat{\sigma}_k. \qquad \ldots (3.50b)$$

$$\hat{\sigma}_3 * \mid \hat{b} > \pm \Delta^{1/2} \mid \hat{b} >,\ \ \hat{\sigma}^{\hat{2}} * \mid \hat{b} > = 3\,\Delta \mid \hat{b} >,. \qquad \ldots (3.50c)$$

(2) *Irregular isopauli matrices*

$$\hat{\sigma}_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \hat{\sigma}_2' = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2, \hat{\sigma}_3' = \begin{pmatrix} g_{22} & 1 \\ 1 & -g_{11} \end{pmatrix} = \Delta\hat{I}\sigma_3, \quad \text{... (3.51a)}$$

$$[\hat{\sigma}'_1, \hat{\hat{\sigma}}_2']_{\hat{\xi}} = 2i\hat{\sigma}_3', \quad [\hat{\sigma}_2', \hat{\sigma}_3']_{\xi} = 2i\,\Delta\,\hat{\xi}_1', \quad [\hat{\sigma}_3', \hat{\sigma}_1']_{\hat{\xi}} = 2i\hat{\sigma}_2', \quad \text{... (3.51b)}$$

$$\hat{\sigma}_3' * |\hat{b}> = \pm\Delta|\hat{b}>, \quad \hat{\hat{\sigma}}^{\hat{2}} * |\hat{b}> = \Delta(\Delta+2)\,|\,\hat{b}>. \quad \text{... (3.15c)}$$

(3) *Standard isopauli matrices*

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & \lambda \\ 0 & -\lambda \end{pmatrix}, \quad \text{... (3.52a)}$$

$$T = \text{diag.}\,(\lambda, \lambda^{-1}), \ \lambda \neq 0, \ \Delta = \det T = 1, \ [\hat{\sigma}''_i, \hat{\sigma}''_j]_{\hat{\xi}} = i \in_{ijk} \hat{\sigma}''_k, \quad \text{... (3.52)}$$

The primary differences in the above isorepresentations are the following. For the case of the regular isorepresentations, the isotopic contributions can be factorized with respect to the conventional Lie spectrum. For the iregular case this is no longer possible. Finally, for the standard case we have conventional spectra of eigenvalues under a generalized structure of the matrix representations, as indicated by the appearance of a completely unrestricted, integro-differential function λ.

The regular and irregular representations of $\hat{o}(3)$ and $s\hat{u}(2)$ are applied to the angular momentum and spin of particles under extreme physical conditions, such as an electron in the core of a collapsing star. The standard isorepresentations are applied to conventional particles evidently because of the preservation of conventional quantum numbers The appearance of the isotopic degrees of freedom then permit novel physical applications, that is, applications beyond the capacity of Lie's theory even for the simpler case of preservation of conventional spectra (see Section 3.*G*.)

The spectrum-preserving map from the conventional representations J_g of a Lie-algebra L with metric tensor g to the covering isorepresentations $\hat{\mathcal{J}}_{\hat{g}}$ of the Lie-Santilli algebra $\hat{L}$ with isometric $\hat{g} = Tg$ and isounit $\hat{I} = T^{-1}$ is important for physical application. It is called the *klimyk rule* [27] and it is given by

$$\hat{\mathcal{J}}_{\hat{g}} = J_g\,p, \ \ p = k\hat{I}, \ \ k \in \hat{F}, \quad \text{... (3.53)}$$

under which Lie algebras are turned into Lie-Santilli isoalgebras

$$J_i J_j - J_j J_i = C^k_{ij} J_k \equiv (\hat{\mathcal{J}}_i * \hat{\mathcal{J}}_j - \hat{\mathcal{J}}_j * \hat{\mathcal{J}}_i)k^{-1}\,T = C^k_{ij}\,k^{-1}\,T\hat{\mathcal{J}}_k,$$

that is,

$$\mathfrak{J}_i * \mathfrak{J}_j - \mathfrak{J}_j * \mathfrak{J}_i = C^k_{ij} \mathfrak{J}_k,$$

thus showing the preservation of the original structure constants.

However, by no means, the Klimyk rule can produce *all* Lie-Santil isoalgebras, because the latter are generally characterized by nounitary transforms of conventional algebras, with a general variation of the structure constants.

Nevertheless, the Klimyk rule is sufficient for a number of physical applications where the preservation of conventional quantum numbers is important, because it permits the identification of one specific and explicit form of standard isorepresentations with "hidden" degrees of freedom represented by the isotopic element T available for specific uses. For instance, the standard isopauli matrices permit the reconstruction of the exact isospin symmetry in nuclear physics under electromagnetic and weak interactions [63], or the construction of the isoquark theory with all conventional quantum numbers, yet an *exact confinement* (with an identically null probability of tunnel effects for free quarks because of the incoherence between the interior and exterior Hilbert spaces) [68], and other novel applications.

3.F. Isotopies and isodualities of the Lorentz and Poincare' symmetries. Consider the line element in Minkowski space $x^2 = x^\mu \eta_{\mu\nu} x^\nu$, $\mu, \nu = 1, 2, 3, 4$, with local coordinates $x = \{x^1, x^2, x^3, x^4\}$, $x^4 = c_0 t$, and metric $\eta =$ diag. $(1, 1, 1, -1)$. Its simple invariance group, the six-dimentsional Lorentz group L (3.1), is characterized by the (ordered sets of) parameters given by the Euler's angles and speed parameter, $w = \{w_k\} = \{\theta, v\}$, $k = 1, 2, ..., 6$, and generators $X = \{X_k\} = \{M_{\mu\nu}\}$ in their known fundamental representation (see, e.g., [31], [32].

Suppose now that the Minkowskian line element is lifted into the most general possible nonlinear-integral form verifying the conditions of Class III

$$x^{\hat{2}} = x^\mu \hat{g}_{\mu\nu} (x, \dot{x}, \ddot{x}, ...) x^\nu, \quad \det \hat{g} \neq 0, \quad \hat{g} = \hat{g}^\dagger, \qquad ... (3.54)$$

which represent: all modifications of the Minkowski metric as encountered, e.g., in particle physics; conventional exterior gravitational line elements with $\hat{g} = \hat{g}(x)$, such as the full Schwartzschild line element; all its possible generalizations for the interior problem; etc.

The explicit form of the simple, six-dimensional invariance of generalized line element $x^{\hat{2}}$ was first constructed by Santilli [51] by following the space-time version of Steps 1 to 5 of the preceding section. Step 1 is the identification of the fundamendal isotopic element T via the factorization of the Minkowski metric $\hat{g} = T\eta$ which, under the assumed conditions, can always be diagonalized into the form

$$T = \text{diag.}\ (g_{11}, g_{22}, g_{33}, g_{44}),\ T = T^{\tau},\ \det T \neq 0. \quad \text{... (3.55)}$$

The fundamental isounit of the theory is then given by $\hat{I} = T^{-1}$.

Step 2 is the lifting of the conventional numbers into the isonumbers via the isofields $\hat{\mathcal{R}}(\hat{n}, +, *)$, $\hat{n} = n\hat{I}$ (which are different than those of $\hat{O}(3)$ because of the different dimension of the isounit).

Step 3 is the construction of the isospaces in which the isometric $\hat{g}$ is properly defined, which are given by the isominkowski spaces $\hat{M}(x, \hat{g}, \hat{\mathcal{R}})$. The reader should keep in mind that, when $\hat{g}$ is a conventional Riemannian metric, isospaces $\hat{M}(x, \hat{g}, \hat{\mathcal{R}})$ are not the Riemannain spaces $R(x, \hat{g}, \mathcal{R})$ because the basic units of the two spaces are different.

Step 4 is also straightforward. The *Lorentz-Santilli isosymmetry* $\hat{L}(3.1)$ is characterized by the isotransformations

$$\hat{O}(3.1)\ .\quad x' = \hat{\Lambda}(\hat{w}) * x = \bar{\Lambda}(w)x, \quad \text{... (3.56)}$$

verifying the basic properties

$$\hat{\Lambda}^{\tau} = \hat{\Lambda}\,\hat{g}\,\hat{\Lambda}^{\tau} = \hat{I}\hat{g}\hat{I},\quad or\quad \bar{\Lambda}\,\tau\,\hat{g}\,\bar{\Lambda} = \bar{\Lambda}\,\hat{g}\,\bar{\Lambda}\tau = \hat{g}, \quad \text{... (3.57a)}$$

$$\hat{\text{Det}}\ \hat{\Lambda} = [\text{Det}\ (\hat{\Lambda}\ T] = \pm\,\hat{I}. \quad \text{... (3.57b)}$$

It is easy to see that $\hat{L}(3.1)$ preserves the original connectivity properties of $L(3.1)$ (see [61] for a detailed study). The connected component $S\hat{O}(3.1)$ of $\hat{L}(3.1)$ is characterized by $\hat{\text{Det}}\ \hat{\Lambda} = +\hat{I}$ and has the structure [loc. cit.]

$$\hat{\Lambda}(w) = \Pi^*_{k\,=\,1,\,2,\,...,\,6}\ e^{ix^*_k\,\hat{w}_k} = \{\prod\nolimits_{k\,=\,1,\,2,\,...,\,6} e^{iX_k\,Tw_k}\}\hat{I} \quad \text{... (3.58)}$$

where the paramenters are the conventional ones, the generators X_k are also the conventional ones in their fundamental representation and the isotopic element T is given by Equations (3.23). The discrete part of $\hat{L}(3.1)$ is characterized by $\hat{\text{Det}}\ \hat{\Lambda} = -\hat{I}$, and it is given by the space-time isoinversions [loc . cit.]

$$\hat{\pi} * x = \pi\, x = -\,r, x^4),\quad \hat{T} * x = Tx = (r, -x^4). \quad \text{... (3.59)}$$

Again, under the assumed conditions for T, the convergence of infinite series (3.58) is ensured by the original convergence, thus permitting the explicit calculation of the symmetry transformations in the needed explicit, finite form. Their space components have been given in the preceding Section 3.E. The additional *Lorentz-Santilli* isoboosts can also be explicitly computed, yielding the expression for all possible isometrics $\hat{g}$[51]

$$x'^1 = x^1,\ x'^2 = x^2, \quad \text{... (3.60a)}$$

$$x'^3 = x^3 \cosh\,[v\,(g_{33}\,g_{44})^{1/2}] - x^4\,g_{44}(g_{33}\,g_{44})^{-\,1/2} \sinh\,[v(g_{33}\,g_{44})^{1/2}] =$$

$$= \hat{\gamma}(x^3 - g_{33}^{-\,1/2}\,g_{44}^{1/2}\,\hat{\beta}x^4), \quad \text{... (360b)}$$

$$x'4 = -x^3 g_{33}(g_{33}\, g_{44})^{-1/2} \sinh\,[v\,(g_{33}\, g_{44})^{1/2}] + x^4 \cosh\,[v\,(g_{33}\, g_{44})^{1/2}] =$$
$$= \hat{\gamma}\,(x^4 - g_{33}^{1/2}\, g_{44}^{-1/2}\, \hat{\beta}\, x^3), \qquad \text{... (3.60c)}$$

where

$$x^4 = c_0 t,\ \ \beta = v/c_0,\ \ \beta = v^k\, g_{kk}\, v^k / c_0\, g_{44}\, c_0, \qquad \text{... (3.61a)}$$

$$\cosh\,[v(g_{33}\, g_{44})^{1/2}] = \hat{\gamma} = |\,1 - \hat{\beta}^2\,|^{-1/2},\ \sinh\,[v(g_{33}\, g_{44})^{1/2}] = \hat{\beta}\hat{\gamma}. \qquad \text{... (3.61b)}$$

Again, one should note : (A) the unrestricted character of the functional dependence of the isometric $\hat{g}$; (*B*) the remarkable simplicity of the final results whereby the explicit symmetry transformations are merely given by plotting the values $g_{\mu\mu}$ in Equations (3.60); and (C) the generally nonlinear-nonlocal-noncanonical character of the isosymmetry.

The isocommutation rules when the generators $M_{\mu\nu}$ are in their regular representation can also be readily computed and are given by [loc. cit]

$$\hat{O}(3.1) : [M_{\mu\nu}\,\hat{,}\,M_{\alpha\beta}] = \hat{g}_{\nu\alpha}\, M_{\beta\mu} - \hat{g}_{\mu\alpha}\, M_{\beta\nu} - \hat{g}_{\nu\beta}\, M_{\alpha\mu} + \hat{g}_{\mu\beta}\, M_{\alpha\nu}\,, \qquad \text{... (3.62)}$$

with isocasimirs

$$C^{(0)} = \hat{I},\ C^{(1)} = \tfrac{1}{2}\, M_{\mu\nu}\, T\, M^{\mu\nu} = M * M - N * N, \qquad \text{... (3.63a)}$$

$$C^{(3)} = \tfrac{1}{2} \in^{\mu\nu\rho\sigma} M_{\mu\nu}\, T\, M_{\rho\sigma} = -M * N,\ M = \{M_{12}, M_{23}, M_{31}\},\ N = \{M_{01}, M_{02}, M_{03}\} \qquad \text{... (3.63c)}$$

The classification of all possible isotopes $S\hat{O}(3.2)$ was also done in the original construction [51] via the realizations of the isotopic element

$$T = \text{diag.}\ (\pm\, b_1^2, \pm\, b_2^2, \pm\, b_3^2, \pm\, b_4^2),\ \ b_\mu > 0,\ \ \mu = 1, 2, 3, 4, \qquad \text{... (3.64)}$$

where the *b's* are the characteristic functions of the interior medium, resulting in:

(1) The conventional orthogonal symmery $SO(4)$ for $T = \text{diag.}\ (1, 1, 1, -1)$;

(2) The conventional Lorenty symmery $SO(3.1)$ for $T = \text{diag.}\ (1, 1, 1, 1)$;

(3) the conventional de Sitter symmetry $SO(2.2)$ for $T = \text{diag.}\ (1, 1, -1, 1)$;

(4) the isodual $SO^d(4)$ for $T = \text{diag.}\ (-1, -1, -1, 1)$;

(5) the isodual $O^d(3.1)$ for $T = -\,\text{diag.}\ (1, 1, 1, 1)$;

(6) the isodual $SO^d(2.2)$ for $T = \text{diag.}\,(-1, -1, 1, -1)$;

(7) the infinite family of isotopes $S\hat{O}(4) \approx SO(4)$ for

$$T = \text{diag.}\,(b_1^2, b_2^2, b_3^2, -b_4^2);$$

(8) the infinite family of isotopes $S\hat{O}(3.1) \approx SO(3.1)$ for

$$T = \text{diag.}\,(b_1^2, b_2^2, b_3^2, b_4^2);$$

(9) the infinite family of isotopes $S\hat{O}(2.2) \approx SO(2.2)$ for

$$T = \text{diag.}\,(-b_1^2, b_2, b_3^2, b_4^2);$$

(10) the infinite family isoduals $S\hat{O}^d(4) \approx SO^d(4)$ for

$$T = \text{diag.}\,(-b_1^2, -b_2^2, -b_3^2, b_4^2);$$

(11) the infinite family of isoduals $S\hat{O}^d(3.1) \approx SO(3.1)$ for

$$T = -\,\text{diag.}\,(b_1^2, -b_2^2, -b_3^2, b_4^2);$$

(12) the infinite family of isoduals $S\hat{O}^d(2.2) \approx SO^d(2.2)$ for

$$T = \text{diag.}\,(b_1^2, -b_2^2, -b_3^2, -b_4^2).$$

On the basis of the above results, Santilli [61] submitted the *conjecture that all simple Lie algebra of the same dimension over a field of characteristic zero in Cartan classification can be unified into one single abstract isotopic algebra of the same dimension.*

The above conjecture was proved by Santilli for the cases $n = 3$ and 6. A theorem unifying all possible fields into the isoreals was proved by Kadeisvili et al [26] in the expectation of such general unification, but its study has remained unexplored at this writing.

In the above presentation we have shown that the lifting of the Lorentz symmetry can be naturally formulated for Class III. Nevertheless, whenever dealing with physical applications, the isotopic element is restricted to have the positive-or negative- definite structure $T = \pm\,\text{diag.}\,(b_1^2, b_2^2, b_3^2, b_4^2)$, thus restricting the isotopies $S\hat{O}(3.1) \approx SO(3.1)$ and $S\hat{O}^d(3.1) \approx SO^d(3.1)$.

The operator realization of the latter Lorentz-Santilli is the following. The linear four-momentum admits the isotopic realization

$$p_\mu * \mid \hat{\psi}> = -i\,\hat{\partial}_\mu \mid \hat{\psi}> = -iT_\mu^\nu\,\partial\nu \mid \hat{\psi}>.$$

Also for $x_\mu = \eta_{\mu\nu}x^\nu$ (where η is the conventional Minkowski metric). one can show that $\hat{\partial}_\mu x_\nu = \hat{\eta}_{\mu\nu}$. The fundamental relativistic isocommutation rules are then given by [61], [65]

$$[x_\mu \,\hat{,}\, p_\nu] = i\,\hat{\eta}_{\mu\nu}, \quad [x_\mu \,\hat{,}\, x_\nu] = [p_\nu \,\hat{,}\, p_\nu] = 0.$$

The isocommutation rules are then given by

$$\hat{O}(3.1) : [M_{\mu\nu} \hat{,} M_{\alpha\beta}] = i(\hat{\eta}_{\nu\alpha} M_{\beta\mu} - \hat{\eta}_{\mu\alpha} M_{\beta\nu} - \hat{\eta}_{\nu\beta} M_{\alpha\mu} + \hat{\eta}_{\mu\beta} M_{\alpha\nu}), \quad \text{... (3.62)}$$

thus confirming the isomorphism $S\hat{O}(3.1) \approx SO(3.1)$ for all positive-definite T.

The Poincare′-Santilli isosymmetry

$$\hat{p}(3.1) = \hat{L}(3.1) \times \hat{T}(3.1), \quad \text{... (3.65)}$$

and its isodual $\hat{p}^d(3.1)$ have been constructed in their classical [62] and operator [62] forms as well as in their isospinorial form $\hat{\mathcal{P}}(3.1) = S\hat{L}(2.\hat{C}) \times \hat{T}(3.1)$ [69]. We here limit ourselves to a brief outline of the nonspinorial case mainly to illustrate the advances in the structure of isoalgebras and isogroups studied in this paper.

A generic element of $\hat{P}(3.1)$ can be written $\hat{A} = (\hat{\Lambda}, \hat{a})$, $\hat{\Lambda} \in \hat{O}(3.1)$, $\hat{a} \in \hat{T}(3.1)$ with isocomposition

$$\hat{A}' * \hat{A} = (\hat{\Lambda}', \hat{a}') * \hat{\Lambda}, \hat{a}) = (\hat{\Lambda} * \hat{\Lambda}', \hat{a} + \hat{\Lambda}' * a'), \quad (3.66)$$

The realization important for physical application is that via conventional generators in their adjoint representation for a system of n particles of non-null mass m_a

$$X = \{X_k\} = \{M_{\mu\nu} = \Sigma_m (x_{a\mu} p_{a\nu} - x_{a\nu} p_{a\nu}\}, p = \Sigma_a p_a\}, k = 1, 2, ..., 10, \quad \text{... (3.67)}$$

and conventional parameters $w = \{w_k\} = \{v, \theta, a\}$, where v represents the Lorentz parameters, θ represents the Euler's angles, and a characterizes conventional space-time translations.

The connected component of the isopoincare′ group is given by

$$\hat{p}(3.1) : x' = \hat{A} * x, \hat{A} = \prod_k^* e_\xi^{iX_k w_k} = \{\prod_k e^{iX_k T w_k}\}\hat{I}, \quad \text{... (3.68)}$$

where the isotopic element T and the Lorentz generators $M_{\nu\mu}$ have the same realization as for $\hat{O}(3.1)$. The primary different with isosymmetries $\hat{O}((3.1)$ is the appearance of the isotranslations.

$$\hat{T}(3.1) * x = \{e_\xi^{iP\,\eta a}\} * x = e_\xi^{iP\hat{g}a}\} * x = x + \hat{a}, \ \hat{T}(3.1) * p \equiv 0. \quad \text{... (3.69)}$$

The general Poincare′-Santilli isotransformations are then given by [61], [62]

$$x' = \hat{\Lambda} * x \text{ Lorentz-Santilli isotransforms,} \quad \text{... (3.70a)}$$

$$x' = x + \hat{a} = x + a_0 B(s, x, \dot{x}, \ddot{x}, ...), \text{ isotranslations,} \quad \text{... (3.70b)}$$

$$x' = \Lambda x' = \hat{\pi}_r * = (-r, x^4), \text{ space isoinversions,} \quad \text{... (3.70c)}$$

$$x' = \hat{\pi}_t * x = (r, -x^4), \quad \text{time isoinversions,} \quad \text{... (3.70d)}$$

where the B-functions are given by the expansions

$$B_{\mu} = b_{\mu} + a^{\alpha}\, [b_{\mu}, \hat{} P\alpha[/1\,! + a^{\alpha}\, a^{\beta}\, [[b_{\mu}, \hat{}P_{\alpha}], \hat{}P_{\beta}]/2\,! + \ldots \qquad \ldots (3.71)$$

The isocommutation rules of $\hat{P}(3.1)$ in the operator realizations indicated earlier are

$$[M_{\mu\nu}, \hat{}M_{\alpha\beta}] = i(\hat{\eta}_{\nu\alpha}\, M_{\beta\mu} - \hat{\eta}_{\mu\alpha}\, M_{\beta\nu} - \hat{\eta}_{\nu\beta}\, M_{\alpha\mu} + \hat{\eta}_{\mu\beta}\, M_{\alpha\nu}),$$

$$\ldots (3.72a)$$

$$[M_{\mu\nu}, \hat{}P_{\alpha}] = i(\hat{\eta}_{\mu\alpha}\, P_{\nu} - \hat{\eta}_{\nu\alpha}\, P_{\mu}),\ [P_{m}, \hat{}P_{\nu}] = 0,\ \mu, \nu, \alpha, \beta = 1, 2, 3, 4,$$

$$\ldots (3.72b)$$

and the isocenter is characterized by the isocasimirs

$$C^{(0)} = \hat{I},\ C^{(1)} = p^{\hat{2}} = P\,T\,P = P_{\mu}\, \hat{g}^{\mu\nu}\, P_{\nu}, \qquad \ldots (3.73a)$$

$$C^{(2)} = \hat{W}^{\hat{2}} = \hat{W}_{\mu}\, \hat{g}^{\mu\nu}\, \hat{W}_{\nu},\ \ \hat{W}_{\mu} = \varepsilon_{\mu\alpha\beta\rho}\, \hat{J}^{\alpha\beta} * P^{\rho}. \qquad \ldots (3.73b)$$

The *restricted isotransformations* occur when the isotopic element T is constant.

An important application of the isotranslation is the characterization of the so-called *isoplane-waves on* $\hat{M}(x, \hat{\eta}, \hat{\mathfrak{R}})$

$$\hat{\psi}(x) = e_{\xi}^{ipx} = \hat{I}e^{ipTx} = \hat{I}\, e^{ip\mu\, \hat{g}^{\mu\nu}\, x_{\nu}} = \hat{I}e^{i(p_k b_k^2 x_k - p_4 b_4^2 x_4)}, \qquad \ldots (3.74)$$

which are solutions of the isotopic field equations, represents electromagnetic waves propagating within inhomogeneous and anisotropic media such as out atmosphere and offer quite intriguing predictions for experimentally verifiable <novel> effects, that is, effects beyond the predictive or descriptive capacities of the Poincare′ symmetry (see the companion paper [60]).

As one can see, the verification of total convservation laws (for a system assumed as isolated from the rest of the universe), is intrinsic in the very structure of the isosymmetry. In fact, the generators are the conventional ones and, since they are invariant under the action of the group they generate, they characterize conventional total consvervation laws. The simplicity of reading off the total conservation laws from the generators of the isosymmetry should be compared with the rather complex proof in conventional gravitational theories.

The isodual *Poincare′-Santilli isosymmetry* $\hat{P}^{d}(3.1)$ is characterized by the isodual generators $X_k^d = -X^k$, the isodual parameters $w_k^d = -w_k$, and the isodual isotopic element $T^d = -T$, resulting in the change of sign of isotransforms. This impies a novel *law of universal invariant under isoduality* which essentially state that any system which is invariant under a given symmetry is automatically invariant under its isodual. In turn, this law apparently permits novel advances in the study of antiparticles [61].

The significance of the Lie-santilli isotheory for gravitation is illustated by the following important property of the isosymmetry $\hat{P}(3.1)$ which evidently follows from of Theorem 3.5:

Theorem 3.6 [51]. *The Poincare′-Santilli isosymmetry* $\hat{P}(3.1)$ *is directly universal for all infinitely possible (3+1) - dimensional invariants*

$$(x-y)^{\mu}\, \hat{\eta}_{\mu\nu}\,(x, \dot{x}, \ddot{x}, \ldots)\,(x-y)^{\nu}, \quad \hat{\eta} = T\eta, \qquad \ldots (3.75)$$

Note that the above theorem includes as particular cases the conventional Riemannian metric $g(x) = \hat{\eta}(x)$, thus providing the universal invariance of *exterior* gravitation in vacuum. More generally, the theorem includes all infinitely possible signature-preserving modifications of the Minkowski and Riemannian metrics for interior problems. The simplcity of this universal invariance should also be kept in mind and compared with the known complexity of other approaches to nonlinear symmetries. In fact, one merely *plots* the $g_{\mu\mu}$ elements in isotransforms (3.45), (3.60), (3.70) without any need to compute anything, because the invariance of general separation (3.75) is ensured by the theorem. For numerous examples, see [61], [62].

As anticipated in Sect. *I.E.* a remarkable property of the Lie-Santilli theory is the capability to unify in one, single, abstract isosymmetry $\hat{P}(3.1)$ all possible linear or nonlinear, local or nonlocal, Hamiltonian or nonhamiltonian, relativistic or gravitational, exterior and interior, classical and operator systems.

3.G : Mathematical and physical applications. Lie's theory is known to be at the foundation of virtually all branches of mathematics. The existence of intriguing and novel applications in mathematics originating from the Lie-Santilli theory is then self-evident.

With the understanding that mathematical studies are at their first infancy, the isotopies have already identified new branches of mathematics besides isoalgebras, isogroups and isorepresentations. We here mention: the new branch of number theory dealing with isonumbers; the new branch of functional isoanalysis dealing with T-operator special isofunctions, isotransforms and isodistribution; the new branch of topology dealing with the peculiar integro-differential topology of the isotopic theory; the new branch of the theory of manifold dealing with isomanifolds and their intriguing properties; and so on. It is hoped that interested mathematicians will contribute to these novel mathematical advances which have been identified and developed until now solely by physicists.

Lie's theory in its traditional linear-local-canonical formulation is also known to be at the foundation of all branches of contemporary physics. Profound physical implications due to the covering, nonlinear-nonlocal-noncanonical Lie-Santilli theory cannot therefore be dismissed in a credible way.

With the understanding that these latter applications too are at the beginning and so much remains to be done, let us recall the following applications of the Poincare'-Santilli isosymmetry $\hat{P}(3.1)$ (see [61] and [62] for details):

(1) The universal invariance of all possible *conventional* gravitation [51].

(2) The geometric unification of the special and general relativities. In fact, the abstract isotope $\hat{P}(3.1)$ unifies the isosymmetry with gravitational isounit $\hat{I} = [T(x)]^{-1}$, $g(x) = T(x)\eta$, and the realization with isounit $I = \text{diag.}\,(1, 1, 1, 1)$ characterizing the special relativity [51].

(3) The universal invariance for all possible interior extensions of relativistic and gravitational theories [51].

(4) Reconstruction at the isotopic level of the *exact* SU(2)- isospin *symmetry under electromagnetic and weak interactions via* the use of the standard isopauli matrices (3.52) with $\lambda^2 = m_p/m_n$ [63];

(5) Quantitative representation of Rauch's interferometric measures on the 4π –spinorial symmetry via the isotopies of Dirac's equation invariant under $\hat{\mathcal{P}}(3.1)$ [69];

(6) First numerical representation of the total magnetic moment of few-body nuclei via the $S\hat{O}(3)$ symmetry and its direct representation of the deformation of the charge distribution of nucleous and consequential alteration of their intrinsic magnetic moments [69];

(7) Nonlocal representation of the Bose-Einstein correlation from first isotopic principles in full numerical agreement with the data from the UAI experiments, while permitting a causal description of nonlocal interactions and the reconstruction of their exact Poincare symmetry at the isotopic level ([58], [8]);

(8) Quantitative representation of the electron pairing in superconductivity [1];

(9) Quantitative-numerical representation of the behaviour of the meanlives of unstable hadrons with speed (which, as well know, are anomalous between 30 and 100 GeV and conventional between 100 and 400 GeV for the K^0-system) via the isominkowskian geometrization of the physical medium in their interior ([6], [7]);

(10) Application to quarks theories via Klimyk rule for the standard isorepresentations of $S\hat{O}(3)$ with conventional quantum numbers with exact confinement of quarks (permitted by the incoherence of the interior isohilbert and exterior Hilbert spaces), and other intriguing possibilities, such as the regaining of convergent perturbative series for strong interactions (which is possible whenver $|T| \ll 1$) [68];

(11) Numerical representation of Arp's measures on quasars redshift as being due to the decrease of speed of light in chromospheres and its isominkowskian geometrization [37].

(12) Numerical representation of the joint redshift and blushifts of pairs of quasars, particularly when proved via gamma spectroscopy to be physically connected to the associated galaxies, and prediction of a measureable isominkowskian redshift for sunlight at sunset [67].

(13) Application to local realism via the proof that Bell's inequality,von Neumann's theorem and all that are inapplicable (rather than "violated") under isotopies (evidently because of the nounitary Structure of the lifting), thus permitting an isotopic completion of quantum mechanics much along the celebrated *E-P-R* argument [65];

(14) Application to q-deformations, discrete time theories and other ongoing studies via their axiomatization into a form invariant under their own time evolution and which coincide with the conventional quantum mechanical axiomatization at the abstract level [33]; and other applications (see monographs [61] and [62];

(14) Novel possibilities in theoretical biology, such as a quantitative representation of the growth of sea shells which, according to computer simulations, crack during their growth is subjected to the conventional Minkowskian geometry, while admit a normal growth under the covering isominkowskian geometry of Class III (the letter one being needed to represent bifurcations which require inversions of time [60].

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LAURICELLA'S FUNCTIONS AND THEIR TRANSFORMATIONS. I

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ABSTRACT

In this paper we shall obtain a transformation for Lauricella's function with the help of difference operators.

1. INTRODUCTION.

It is well known that difference operators Δ and E play a very impotant role in the field of special functions. Agarwal [1, 2, 3] and Gupta and Agarwal [5, 6, 7] have successfully applied these operators to obtain various tranformations and also evaluated several integrals in a very simple manner. In this paper we shall obtain the transformation [2, p. 116]

$$F_A[a; b_1, b_2, ..., b_n; c_1, c_2, ..., c_n; x_1, x_2, ..., x_n]$$

$$= (1 - x_1 - ... - x_k)^{-a} F_A[a; c_1 - b_1, c_2 - b_2, ..., c_k - b_k, b_{k+1},$$

$$b_{k+2}, ..., b_n; c_1, ..., c_n ; \frac{-x_1}{1 - x_1 - ... - x_k}, ...,$$

$$\left.\frac{-x_k}{1 - x_1 - ... - x_k}, \frac{x_{k+1}}{l - x_1 - ... - x_k}, ..., \frac{x_n}{1 - x_1 - ... - x_k}\right] \quad (*)$$

for Lauricella's functions with the help of difference operators.

2. FIRST STEP OF PROOF

We know that

$$F_2(a; b_1, b_2; c_1, c_2; x_1, x_2]$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(a)_{m_1+m_2} (b_1)_{m_1} (b_2)_{m_2} x_1^{m_1} x_2^{m_2}}{(c_1)_{m_1} (c_2)_{m_2} m_1! \, m_2!}$$

$$= \frac{\Gamma(c_1)\,\Gamma(c_2)}{\Gamma(b_1)\ \Gamma(b_2)} (1 - x_1E_1 - x_2E_2)^{-a} \frac{\Gamma(b_1)\,\Gamma(b_2)}{\Gamma(c_1)\ \Gamma(c_2)},$$

where $E_1^n f(\alpha_1) = f(\alpha_1 + n)$ and E_i operates on b_i and c_i only such that

$$\frac{\Gamma(c_i)}{\Gamma(b_i)} E_i^n \left(\frac{\Gamma(b_i)}{\Gamma(c_i)}\right) = \frac{(b_i)_n}{(c_i)_n} \quad .... (**)$$

This suggests a generalization of this method i.e., consider

$$A = \left[\frac{\Gamma(c_1) \dots \Gamma(c_n)}{\Gamma(b_1) \dots \Gamma(b_n)}\right] . (1 - x_1 E_1 - \dots - x_n E_n)^{-a} \left[\frac{\Gamma(b_1) \dots \Gamma(b_n)}{\Gamma(c_1) \dots \Gamma(c_n)}\right]. \quad \dots (1)$$

Now we shall show with the help of following Lemmas that this general form reduces to Lauricella's function F_A.

Lemma 1. It is well known that (multinomial Theorem)

$$(x_1 + \dots + x_n)^k = \sum_{\Sigma m_i = k} \frac{k!\, x_1^{m_1} \dots x_n^{m_n}}{m_1! \dots m_n!}.$$

Lemma 2 (Srivastava. [8, p. 4]).

$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} (x_1 + \dots + x_n)^k$$

$$= \sum_{m_1=0}^{\infty} \dots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1 + \dots + m_n}}{m_1! \dots m_n!} x_1^{m_1} \dots x_n^{m_n}.$$

It can be easily proved either by induction method [8, p. 4] or with the help of Lemma 1.

Now consider

$$A = \left[\frac{\Gamma(c_1) \dots \Gamma(c_n)}{\Gamma(b_1) \dots \Gamma(b_n)}\right]$$

$$.(1 - x_1 E_1 - \dots - x_n E_n)^{-a} \left[\frac{\Gamma(b_1, \dots \Gamma(b_n)}{\Gamma(c_1) \dots \Gamma(c_n)}\right],$$

$$= \left[\frac{\Gamma(c_1) \dots \Gamma(c_n)}{\Gamma(b_1) \dots \Gamma(b_n)}\right] \sum_{k=0}^{\infty} \frac{(a)_k}{k!} .(x_1 E_1 + \dots + x_n E_n)^k \left[\frac{\Gamma(b_1) \dots \Gamma(b_n)}{\Gamma(c_1) \dots \Gamma(c_n)}\right].$$

In view of Lemma 1 and Lemma 2 we have,

$$A = \left[\frac{\Gamma(c_1) \dots \Gamma(c_n)}{\Gamma(b_1) \dots \Gamma(b_n)}\right] \sum_{m_1=0}^{\infty} \dots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1 + \dots + m_n}}{m_1! \dots m_n!}$$

$$. (x_1 E_1)^{m_1} \dots (x_n E_n)^{m_n} \left[\frac{\Gamma(b_1) \dots \Gamma(b_n)}{\Gamma(c_1) \dots \Gamma(c_n)}\right]$$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}}{m_1!\ldots m_n!} \cdot \frac{(b_1)_{m_1} \ldots (b_n)_{m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}} \cdot x_1^{m_1} \ldots x_n^{m_n}.$$

$$= F_A\,[a; b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n], \qquad \ldots (2)$$

where, for convergence,

$$|x_1| + x_2| + \ldots + |x_n| < 1.$$

It has now been shown that A equals the 1hs of (*).

3. SECOND STEP

To prove that A equals also the rhs of (*) we set $E_i = 1 + \Delta_i$. It can be shown that

$$\frac{\Gamma(c_i)}{\Gamma(b_i)} \Delta_i^m \left(\frac{\Gamma(b_i)}{\Gamma(c_i)} \right) = \frac{(-1)^m (c_i - b_i)_m}{(c_i)_m}. \qquad \ldots (***)$$

Thus,

$$A = \left[\frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_1) \ldots \Gamma(b_n)} \right] (1 - x_1 E_1 - \ldots - x_n E_n)^{-a} \left(\frac{\Gamma(b_1) \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)} \right)$$

$$= \left[\frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_1) \ldots \Gamma(b_n)} \right] (1 - x_1 - \ldots - x_k)^{-a}$$

$$\cdot \left[1 - \frac{x_1 \Delta_1 + \ldots + x_k \Delta_k + x_{k+1} E_{k+1} + \ldots + x_n E_n}{1 - x_1 - \ldots x_k} \right]^{-a} \left(\frac{\Gamma(b_1) \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)} \right)$$

Again using Lemma 1 and Lemma 2 we have,

$$A = \left[\frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_1) \ldots \Gamma(b_n)} \right] (1 - x_1 - \ldots x_k)^{-a}$$

$$\cdot \sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}}{m_1! \ldots m_n!} \left(\frac{x_1 \Delta_1}{1 - x_1 - \ldots - x_k} \right)^{m_1}$$

$$\ldots \left(\frac{x_k \Delta_k}{1 - x_1 \ldots - x_k} \right)^{m_k} \left(\frac{x_{k+1} E_{k+1}}{1 - x_1 - \ldots - x_k} \right)^{m_{k+1}} \ldots$$

$$\left(\frac{x_n E_n}{1 - x_n - \ldots - x_k} \right)^{m_n} \left(\frac{\Gamma(b_1) \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)} \right),$$

Hence, from (1), (2) and (***), we see that A also equals the rhs of (*).

This completes the proof.

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LEFT ALTERNATIVE RINGS WITH AN IDEMPOTENT

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ABSTRACT

Let R be a left alternative ring that satisfies $(x, y, z) + (y, z, x) + (z, x, y) = 0$. An idempotent element in R belongs to the nucleus of R. Let R be a prime ring with characteristic prime to 6. If R has an idempotent $e \neq 0, 1$ then R is associative. We give an example of a ring under consideration with an idempotent $e \neq 0, 1$.

1. INTRODUCTION

A non-associative ring R is called left alternative if

$$(x, y, z) + (y, x, z) = 0 \qquad \dots (1)$$

for all x, y, z in R where the associator $(x, y, z) = (xy)z - x(yz)$. That is,

$$(x, x, z) = 0 \qquad \dots (i)$$

for all x, z in R.

Kleinfed and Smith [1] have shown that a prime left alternative ring with commutators in the left nucleus and characteristic $\neq 2, 3$ is associative. Rich (2) has considered rings with idempotents in their nuclei.

2. PRELIMINARIES

In any arbitrary ring we have

$$f(w, x, y, z) = (wx, y, z) - (w, xy, z) + (2, x, yz) - w(x, y, z) - (w, x, y)z = 0$$

Then using (1) in $f(x, y, y, z) - f(y, y, x, z) + f(y, x, y, z) = 0$

we get $2(xy, y, z) - 2y(x, y, z) = 0$. Assuming that characteristic $\neq 2$,, we have

$$(xy, y, z) = y(x, y, z) \qquad \dots (2)$$

Linearizing (2) we get

$$g(x, w, y, z) = (xw, y, z) + (xy, w, z) - w(x, y, z) - y(x, w, z) = 0.$$

Using (1) and (2) in $f(x, y, y, z) = 0$, we get

$$(x^2, y, z) = (x, xy + yx, z).$$

3. MAIN SECTION

Unless otherwise stated R will be a left-alternative ring that satisfies

$$(x, y, z) + (y, z, x) + (z, x, y) = 0 \quad \ldots (4)$$

for all x, y, z in R.

Lemma 1. Let A be a non-zero ideal of R. Then $B = \{x \in R : xA = Ax = (0)\}$ is an ideal of R.

Proof. Let $a \in A, x \in B$ and $y \in R$. Then

$$0 = (y, a, x) + (a, y\,x) = (ya)x - y(ax) + (ay)x - a(yx).$$

But $a \in A$ implies that $ya, ay \in A$. Therefore, we have $a(yx) = 0$.

So $(a, y, x) = (ay)x - a(yx) = 0$. Now from (4)

$$0 = (y, x, a) + (x, a, y) + (a, y, x) = (y, x, a) + (x, a, y)$$
$$= (yx)a - y(xa) + (xa)y - x(ay) = (yx)a.$$

Therefore $yx \in B$. Interchanging the role of x and y in above we get $xy \in B$. Hence B is an ideal of R.

Now we assume that $((e, x), e, e) = 0$ where e is an idempotent of R and $(e, x) = ex - xe$.

Theorem 2. Let R be a ring of characteristic prime to 6. If R has an idempotent $e \neq 0, 1$ then R has the desired Peirce decomposition $R = R_{11} + R_{10} + R_{01} + R_{00}$ where x belongs to R_{ij} if and only $ex = ix$ and $xe = jx$ for $i, j = 0, 1$ the sum of the sub-mdodules is direct.

Proof. It suffices to show that $(e, e, x) = (e, x, e) = (x, e, e) = 0$ for all x in R. Let $x \in R$. Then

$$(e, x, e) = (e^2, x, e) = (e, ex + xe, e) \quad \text{(by (3))}$$
$$= (e, ex, e) + (e, xe, e)$$

Now $\quad 0 = ((e, x), e, e) = (e, (e, x), e) \quad$ (by (1))

This implies that $(e, ex, e) = (e, xe, e)$. Therefore, $(e, x, e) = 2(e, ex, e)$. Therefore,

$(e, x, e) = 2(e, ex, e)$. That is,

$$(e, x, e) - 2(e, ex, e) = 0 \quad \ldots (5)$$

for all x in R. In particular,

$(e, ex, e) - 2(e, e(ex), e) = 0.$

But since R is left-alternative $(e, e, x) = 0$. Therefore, $e(ex) = (ee)x = ex$.

Thus, $(e, ex, e) = 0$.

Substituting in (5), we get $(e, x, e) = 0$.

Now $(e, x, e) + (x, e, e) + (e, e, x) = 0$ implies

that $(x, e, e) = 0$. Therefore $(e, x\,e) = (x, e\,e) = (e, e, x) = 0$.

Hence R has the desired Peirce decomposition.

Lemma 3. Suppose R has a Peirce decomposition with respect to e. Then

(i) $R_{ii} R_{ii} \subseteq R_{ii}$

(ii) $R_{ij}R_{kl} \subseteq \delta_{jk} R_{il}$ for all $i, j, k, l = 0, 1$.

Proof. Let $x_{11}, y_{11} \in R_{11}$. Then $e(x_{11}\, y_{11}) = -(e, x_{11}, y_{11}) + ex_{11})\, y_{11}$
$= (x_{11}, e, y_{11}) + x_{11}\, y_{11} = (x_{11}\, e)y_{11} - x_{11}(e\, y_{11}) + x_{11}\, y_{11}$
$= x_{11}\, y_{11} - x_{11}\, y_{11} + x_{11}\, y_{11} = x_{11}\, y_{11}$. And

$$\begin{aligned}(x_{11}\, y_{11})e &= x_{11}, y_{11}, e) + x_{11}(y_{11}e)\\ &= -(y_{11}, e, x_{11}) - (e, x_{11}, y_{11}) + x_{11}\, y_{11} \qquad \text{(by (4))}\\ &= -(y_{11}, e)x_{11} + y_{11}(ex_{11}) + (x_{11}, e, y_{11})\, x_{11}\, y_{11}\\ &= -y_{11}\, x_{11} + y_{11}\, x_{11} + (x_{11}e)y_{11} - x_{11}\,(ey_{11}) + x_{11}\, y_{11}\\ &= -y_{11}\, x_{11} + y_{11}\, x_{11} + x_{11}\, y_{11} - x_{11}\, y_{11} + x_{11}\, y_{11} = x_{11}\, y_{11}.\end{aligned}$$

These imply that $x_{11}\, y_{11} \in R_{11}$. Therefore $R_{11}\, R_{11} \in R_{11}$. Similarly other results can also be proved.

Nucleus : For an arbitrary non-associative ring R, the nucleus N is defined by

$$N = \{x \in R : (x, y, z) = (y, z, x) = (y, x, z) = \text{ for all } y, z \in R\}.$$

Lemma 4. Suppose R has an idempotent $e \neq 0, 1$. Then e belongs to nucleus N or R.

Proof. We shall show that $(x, e, y) = 0$ for all x, y in R. This will imply $(e, x, y) = 0$ by (1) and by (4) we get $(x, y, e) = 0$. Let $x, y \in R$ Then $R = R_{11} + R_{10} + R_{01} + R_{00}$ implies that $x = x_{11} + x_{10} + x_{01} + x_{00}$ and $y = y_{11} + y_{10} + y_{01} + y_{00}$ where $x_{ij}, y_{ij} \in R_{ij}$, $i, j = 0, 1$.

Now $(x, e, y) = (xe)y - x(ey) = (x_{11} + x_{01})y - x(y_{11} + y_{10})$
$= x_{11}\, y_{11} + x_{11}\, y_{10} + x_{01}\, y_{11} + x_{01}y_{10} - x_{11}\, y_{11} - x_{01}\, y_{11} - x_{11}\, y_{10} - x_{01}\, y_{10}$
$= 0 (\because R_{ij}R_{kl} = (0)$, if $j \neq k$, $(i, j) \neq (k, l)$ and either $i \neq j$ or $k \neq l$). Hence $e \in N$.

Lemma 5. Suppose y belongs to the nucleus N of R. Then $(y, z) \in N$ for all z in R.

Proof. Let $x, w, z \in R$ and $y \in N$. Then

$$g(x, w, y, z) = (xw, y, z) + (xy, w, z) - w(x, y, z) - y(x, w, z) = 0.$$

implies that $(xy, w, z) = y(x, w, z) = 0$. Changing x to z, w to x and z to w in the above we get

$$(zy, x, w) = y(z, x, w) \qquad \dots (6)$$

Now $f(y, z, x, w) = (yz, x, w) - (y, zx, w) + (y, z, xw) - y(z, x, w) - (y, z, x) = 0$. Since $y \in N$, we have

$$(yz, x, w) = (z, x, w) \qquad \dots (7)$$

From (6) and (7) we get $((y, z)\, x, w) = 0$. By (1) $(x, (y, z), w) = 0$ and by (4) $(x, w, (y, z)) = 0$ for all x, w in R. Hence $(y, z) \in N$.

Definition. A ring R is said to be *prime* if for any two ideals A and B of R, $AB = (0)$ implies $A = (0)$ or $B = (0)$.

Lemma 6. Let R be an arbitrary non-associative prime ring. Then R can contain no non-zero nuclear ideals (ideals in nucleus).

Proof. Let A be an ideal in the nucleus N of R. Let $x, y, z, w \in R$ and $a \in A$. Then

$$f(a, x, y, z) = (ax, y, z) - (a, xy, z) + (a, x, yz) - a(x, y, z) - (a, x, y)z = 0.$$

Since ax , $a \in A \subseteq N$, we get $a(x, y, z) = 0$.

Because $a \in A \subseteq N$, $a((x, y, z)w) = (a(x, y, z))w = 0$.

This implies that $A((R, R, R) + (R, R, R)R) = (0)$. But $(R, R, R) + (R, R, R)R$ is an ideal of R and R is prime. Either $A = (0)$ *or* $(R, R, R) + (R, R, R)R = (0)$. But R is non-associative. THerefore $= (0)$. This proves the lemma.

Lemma 7. Suppose N is the nucleus of R. Then

(*i*) $N(R, R, R) = (NR, R, R)$.

(*ii*) $(R, R, R)N = (R, R, RN)$.

(*iii*) $N(R, R, R) = (R, NR, R)$.

(*iv*) $[N, (R, R, R)] = (0)$.

Proof. Let $n \in N$ and $x, y, z \in R$. Then

(*i*) $f(n, x, y, z) = (nx, y, z) - (n, xy, z) + (n , x, yz) - n(x, y, z) - (n , x, y)z = 0$

implies that $(nx, y, z) = n(x, y, z)$. Therefore $(NR, R, R) = N(R, R, R)$.

(*ii*) $f(x, y, z, n) = (xy, z, n) - (x, yz, n) + (x, y, zn) - x(y, z, n) - (x, y, z)n = 0$

implies that $(x, y, zn) = (x, y, z)n$. Therefore $(R, R, RN) = (R, R, R)N$.

(*iii*) $(x, ny, z) = (ny, x, z) = -n(y, x, z) = n(x, y, z)$

(*iv*) $f(x, n, y, z) = (xn, y, z) - (x, ny, z) + (x, n yz) - x(n, y, z) - (x, n, y) = 0$

Therefore $(R, NR, R) = N(R, R, R)$.

implies that

$$(x, ny, z) = (xn, y, z) \quad \dots (8)$$

By (*ii*)

$$\begin{aligned}
(x, y, z)n &= (x, y, zn) \\
&= -(y, zn, x) - (zn, x, y) && \text{(by (4))} \\
&= (zn, y, x) - (zn, x, y) && \text{(by (1))} \\
&= (z, ny, x) - (z, nx, y) && \text{(by (8))} \\
&= -(ny, z, x) + (nx, z, y) && \text{(by (1))} \\
&= -n(y, z, x) + n(x, z, y) && \text{(by (}i\text{))} \\
&= n((z, y, x) + (x, z, y)) && \text{(by (1))} \\
&= -n(y, x, z) && \text{(by (4))}
\end{aligned}$$

$$= n(x, y, z) \qquad \text{(by (1))}$$

Therefore $[n, (x, y, z)] = 0$. Hence $[N, (R, R, R)] = (0)$.

Corollary 8. $[R, N] \subseteq N$.

Proof. Let $x, y, z \in R$ and $n \in N$. Then using (*i*), (*iii*) and (8), $(nx, y, z) = n(x, y, z) = (x, ny, z) = (xn, y, z)$ Therefore, $((x, n), y, z) = 0$. Using (1), $(y, (x, n), z) = 0$.

Using (4), $(y, z\,(x, n)) = 0$. Hence $(R, N) \subseteq N$.

Lemma 9. Suppose N is the nucleus of R. If I is the ideal generated by the set $[R, N)$, then $I = [R, N] + R[R, N]$.

Proof. Clearly $[R, N] + R[R, N] \subseteq I$. Let $n_1, n_2 \in N$ and $x, y, z, w \in R$. Then $[x, n_1]w = [(x, n_1), w] + w[x, n_1]$ implies that $[x, n_1]\, w \in [R, N] + R[R, N]$ since $(x, n_1) \in (R, N) \subseteq N$. Also, since $[R, N] \subseteq N$, $(y, [z, n_2])w = -y(z, n_2]w) = y[z, n_2)w] + y(w[z, n_2]) \in R[R, N] + R[R, N]$. Therefore $[R, N]$ is a right deal. Because $[R, N] \subseteq N$,

$$R(R, N] + R[R, N]) = R[R, N] + R(R[R, N)) = R[R, N] + (RR)\,[R, N]$$
$$\subseteq R[R, N] + R[R, N] \subseteq [R, N] + R[R, N].$$

Thus $[R, N] + R[R, N]$ is a two sided ideal of R. It contains $[R, N]$ which is contained in I. Hence $I = [R, N] + R[R, N]$.

Lemma 10. In the Peirce decomposition $R_{10}^2 = R_{01}^2 = (0)$.

Proof. Let $x_{10}, y_{10} \in R_{10}$ and $e \in N$. Then $(x_{10}, e, y_{10}) = 0$ implies that $(x_{10}\, e)\, y_{10} = x_{10}(ey_{10})$ or $x_{10} y_{10} = 0$. Similarly considering $(x_{01}, e, y_{01}) = 0$ we can show that $x_{10}\, y_{01} = 0$. Hence $R_{10}^2 = R_{01}^2 = (0)$.

Lemma 11. Suppose R has an idempotent $e \neq 0, 1$. Then the set $B = R_{10}\, R_{01} + R_{10} + R_{01} + R_{01}\, R_{10}$ is an ideal of R contained in the nucleus N of R.

Proof By Lemma 4, $e \in N$. By Lemma 5, $(e, x_{01}) \in N$ for some $x_{01} \in R_{01}$. This impies that $x_{01} \in N$ or $R_{01} \subseteq N$. Again $(e, x_{10}) \in N$ for some $x_{10} \in R_{10}$ implies that $R_{10} \subseteq N$. Since N is a sub- ring of R, we have both $R_{10}\, R_{01}$ and $R_{01}\, R_{10}$ contained in N. Hence $B \subseteq N$. Now using Lemma 2 and the fact that $R_{10}, R_{01} \subseteq N$, we have.

$$RB = (R_{10} + R_{01} + R_{00} + R_{11})\,(R_{10} + R_{10}\,R_{01} + R_{01}\,R_{10} + R_{01})$$
$$\subseteq R_{10}^2 + R_{10}\,(R_{10}\,R_{01}) + R_{10}\,(R_{01}\,R_{10}) + R_{10}\,R_{01} + R_{01}\,R_{10} + R_{01}\,(R_{10}\,R_{01})$$
$$+ R_{01}\,(R_{01}\,R_{10}) + R_{01}^2 + R_{00}\,R_{10} + R_{00}\,(R_{10}\,R_{01}) + R_{00}\,(R_{01}\,R_{10}) + R_{00}\,R_{01}$$
$$+ R_{11}\,R_{10} + R_{11}\,(R_{10}\,R_{01}) + R_{11}\,(R_{01}\,R_{10}) + R_{11}\,R_{01}$$
$$\subseteq O + R_{10}\,R_{11} + R_{10}\,R_{00} + R_{10}\,R_{01} + R_{01}\,R_{10} + R_{01}\,R_{11} + R_{01}\,R_{00}$$
$$+ O + O + R_{00}\,R_{11} + (R_{00}\,R_{01})\,R_{10} + R_{01} + R_{10} + (R_{11}\,R_{10})\,R_{01} + R_{11}\,R_{00} + O$$

$$\subseteq O + O + R_{10} + R_{10}R_{01} + R_{01}R_{10} + R_{01} + O + O + O + O + R_{01}R_{10}$$

$$+ R_{01} + R_{10} + R_{10}R_{01} + O + O \subseteq B$$

Similarly we can show that $BR \subseteq B$. Hence B is an ideal of R contained in N.

Theorem 12. Suppose R is a prime ring of characteristic prime to 6 has an idempotent $e \neq 0, 1$. Then R is associative.

Proof. Suppose R is not associative. Then by lemma 6, R can contain no non-zero nuclear ideal. By lemma 11, $B = (0)$. This implies that both R_{10} and R_{01} are (0) By theorem 2, $R = R_{00} + R_{11}$. Now, by lemma 3, $R_{00}(R_{00}(R_{00} + R_{11}) \subseteq R_{00}R_{00} + R_{00}R_{11} \subseteq R_{00}$ and

$(R_{00} + R_{11})R_{00} \subseteq R_{00}R_{00} + R_{11}R_{00} \subseteq R_{00}$. Therefore, R_{00} is an ideal of R. Similarly R_{11} is an ideal of R. Let $x \in R_{00}$ and $x \in R_{11}$ Then $(x, e, y) = 0$ implies $(xe)y = x(ey)$ which implies $xy = 0$. Therefore $R_{00}R_{11} = (0)$ But R is prime. Either $R_{00} =$ or $R_{11} = (0)$. But $e \neq 0 \in R_{11}$. Therefore, $R_{00} = (0)$. Then $R = R_{11}$. So $ex = xe = x$ for all x in R. Thus $e = 1$. Hence R is associative.

Example 13. Let R be a ring defined by the following multiplication table together with all finite sums of e, a, b

	e	a	b
e	e	0	$-b$
a	b	0	0
b	0	0	0

Note that $(e, a, e) = (ea)e - e(ae) = -eb = b$. Therefore, R is non-associative.

To prove $(x, y, z) + (y, x, z) = 0 \ \forall \ x, y, z \in R$, it suffices to check that

$(e, a, e) + (a, e, e) = 0$.

Now $(a, e, e) = (ae)e = a(ee) = be - ae = 0 - b = -b$

$$\therefore \ (e, a, e) + (a, e, e) = b - b = 0.$$

Also $(e, e, a) = (ee)a - e(ea) = ea = 0$.

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ON A CLASS OF ENTIRE FUNCTIONS REPRESENTED BY TAYLOR DIRICHLET SERIES

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ABSTRACT

In this paper, we have studied certain classes of Taylor Dirichlet series. The standard Taylor series and Dirichlet series turn out to be the particular cases of this series. Various functional analytic structures have been provided to these classes. Topological zero divisors, invertible elements and continuous linear functionals have been investigated for these classes. An interesting method of construction of total sets has also been formulated.

1. Introduction

In (3) Rishishwar has studied the series of type

$$f(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n \psi(z)} \tag{1.1}$$

where,

$$\lim_{n\to\infty} \sup . \frac{n}{\lambda_n} = D < \infty,\ 0 < \lambda_1 < \lambda_2 < \ldots\ (\lambda_n \to \infty \text{ as } n \to \infty).$$

$\{a_n\}$ $(n = 1, 2, 3, \ldots)$ is a sequence of complex numbers $\psi(z)$ is a function of complex variable z.

The standard Taylor series and Dirichlet series turn out to be the particular cases of the series (1.1). Hence, in future we shall call the series (1.1) as Taylor Dirichlet series (TDS-in short).

Suppose that

$$\psi(z) = \phi(-z) + i\,\theta(z)$$

where Φ are real valued functions of the complex variable z and that ψ is invertible and continuous. TDS converges in the half plane (cf. [3]) Re. $\psi(z) = \varphi(z) \le R$. Therefore TDS represents an entire function if and only if

$$\varphi^{-1}\left[\lim_{n\to\infty} \inf. \frac{\log|a_n|}{-\lambda_n}\right] = \infty \tag{1.2}$$

Let E stands for the family of all entire Taylor Dirichlet series (ETDS - in short). The function $f(z), g(z)$ and $h(z)$ are defined as

$$f(z) = \sum_{n=1}^{\infty} a_n^{\lambda_n \psi(z)}$$

$$g(z) = \sum_{n=1}^{\infty} b_n e^{\lambda_n \psi(z)}$$

and
$$h(z) = \sum_{n=1}^{\infty} c_n e^{\lambda_n \psi(z)}.$$

Throughout this paper, summation extends from 1 to ∞, unless limits are specified and f, g, h stands for $f(z), g(z)$ and $h(z)$ respectively.

2. Let F be the subset of E defined by

$$F = \{f(z) : f(z) = \Sigma a_n e^{\lambda_n \psi(z)},\ \sup_{n \geq 1} . e^{\lambda_n \varphi(n)} |a_n| < \infty\}$$

It can easily be seen that every element of F represents a function which converges in the whole complex plane. With pointwise addition and scalar multiplication, F becomes a linear space. The norm in F is defined as follows.

$$\|f\| = \sup_{n \geq 1} e^{\lambda_n \varphi(n)} |a_n|,\ f(z) = \Sigma a_n e^{\lambda_n \psi(z)} \in F.$$

It can easily be varified that F together with this norm defined on it, is a normed linear space. Multiplication in F is defined as

$$f.g = \Sigma e^{\lambda_n \varphi(n)} a_n b_n e^{\lambda_n \psi(z)}, \text{ for every } f, g, \in F.$$

For the definition of terms used we refer [1].

Theorem 1. F is a commutative B^*-algebra with identity.

Algebra- We just note that

$$\begin{aligned} \|f.g\| &= \sup_{n \geq 1} e^{\lambda_n \varphi(n)} \mid e^{\lambda_n \varphi(n)} a_n b_n \mid \\ &\leq \sup_{n \geq 1} e^{\lambda_n \varphi(n)} |a_n| \ \sup_{n \geq 1} . e^{\lambda_n \varphi(n)} |b_n| \\ &= \|f\| \cdot \|g\|. \end{aligned}$$

Identity. The element $e = \Sigma e_n e^{\lambda_n \psi(z)} \in F$, where $e_n = e^{-\lambda_n \varphi(n)}$ serves as identity element in F.

Completeness. Let $\{f_p\}$ be a Cauchy sequence in F, where

$$f_p(z) = \Sigma a_{pn} e^{\lambda_n \psi(z)}$$

Given $\in > 0$ we can find $p_0 \geq 1$ such that

$$\|f_p - f_q\| < \varepsilon, \text{ for every } p, q \geq p_0$$

i.e.

$$\sup_{n \geq 1} e^{\lambda_n \varphi(n)} \, |a_{pn} - a_{qn}| < \varepsilon, \text{ for every, } p\, q \geq p_0.$$

This implies that $\{a_{pn}\}$ is a Cauchy sequence in C for every n and hence, owing to the completeness of C converges to a complex number say, a_n.

Thus

$$f_p \to f, \text{ where } f(z) = \Sigma a_n \, e^{\lambda_n \psi(z)}.$$

Moreover $f(z)$ is a member of F. Since

$$e^{\lambda_n \varphi(n)} \, |a_n| \leq e^{\lambda_n \varphi(n)} \, |a_{pn} - a_n| + e^{\lambda_n \varphi(n)} | \, a_{pn}|$$

Involution. The operation $*$ defined of F by

$$*\,(f) = f*(z) = \Sigma \, \bar{a}_n \, e^{\lambda_n \psi(z)}$$

is an involution mapping. We note that

$$\begin{aligned} \| f.\, f* \| &= \sup_{n \geq 1} .\, e^{\lambda_n \varphi(n)} \, |e^{\lambda_n \varphi(n)} \, a_n \,.\, \bar{a}_n| \\ &= \sup_{n \geq 1} \{e^{\lambda_n \varphi(n)} \, |a_n|\}^2 \\ &= \|f\|^2. \end{aligned}$$

Rest of the proof is straight forward.

Corollary. $(F, \|.\|)$ is a Gelfand Algebra.

Proof. Since $\|e\| = \operatorname{sub}_{n \geq 1} e^{\lambda_n \varphi(n)} \, |\, e^{-\lambda_n \varphi(n)}| = 1$

Theorem 2. A function $f(z) = \Sigma \, a_n \, e^{\lambda_n \psi(z)}$ in F is a topological zero divisor if and only if

(2.1) $$a_n = 0\,(e^{-\lambda_n \varphi(n)})$$

Proof. Consider the sequence $\{g_k\}$ in F, were

$$g_k(z) = e^{\lambda_k [\psi(z)} - \varphi(k)],\ k \geq 1$$

Note that for every $k \geq 1$. $g_k \in F$ and $\|g_k\| = 1$. Moreover,

$$\begin{aligned} f \cdot g_k = g_k \cdot f &= e^{\lambda_k \varphi(k)} \, a_k \,.\, e^{-\lambda_k \varphi(k)} \,.\, e^{\lambda_k \psi(z)} \\ &= a_k \,.\, e^{\lambda_k \psi(z)}. \end{aligned}$$

Hence owing to (2.1)

(2.2) $$\|f \cdot g_k\| = \|g_k \cdot f\| \to 0 \text{ as } k \to \infty$$

Thus $f(z)$ is a topological zero divisor in F. Let on the other hand $f(z) = \Sigma a_n \, e^{\lambda_n \psi(z)}$ be a topological zero divisor in F. There exists therefore a sequence $\{g_k\}$ of elements in F with unit norm such that (2.2) holds.

Let of possible, $\lim_{n \to \infty} e^{\lambda_n \varphi(z)} |a_n| = l \neq 0$. Thus for an arbittary $\gamma > 0$, there exists n_0 such that

(2.3) $$e^{\lambda \varphi(n)} \; |a_n| > l - \gamma, \quad \text{for } n > n_0.$$

By hypothesis, for every $k > 1$

$$\sup_{n \geq 1} e^{\lambda_n \varphi(n)} \; |b_{kn}| = 1.$$

Given $\varepsilon > 0$, we can find for every k; an integer N_k and a subsequence $\{n_i\}$ of indices such that

(2.4) $$e^{\lambda_n \varphi(n)} \, |b_{k \, . \, n_1}| > 1 - \varepsilon \;\; \text{for} \;\; n_i > N_k.$$

(2.3) and (2.4) together imply that

$$\| f.g_k \| > C > 0.$$

Contradicting (2.2). Hence the theorem.

Note that F is not a division algebra. For instance, the element

$$p(z) = \Sigma \lambda_n^{\;-1} \, e^{-\lambda_n \varphi(n)} \, e^{\lambda_n \psi(z)}$$

though belongs to F, does not possess inverse in F. For, if possible, $q(z) = \Sigma d_n \, e^{\lambda_n \psi(z)}$ be its inverse. Hence we must have

$$p \, . \, q = e$$

i.e.

$$e^{\lambda_n \varphi(n)} \, (e^{-\lambda_n \varphi(n)} \, \lambda_n^{-1} \, d_n) = e^{-\lambda_n \varphi(n)}$$

This implies

$$d_n = \lambda_n \, e^{-\lambda_n \varphi(n)}.$$

It can easily be varified that the so defined $q(z) = \Sigma d_n \, e^{\lambda_n \psi(z)}$ is not a member of F. In this connection we propose.

Theorem-3. The function $f(z) = \Sigma \, a_n \, e^{\lambda_n \psi(z)}$is invertible in F if and only if $\{ |e^{\lambda_n \psi(n)} \, a_n|^{-1} \}$ is a bounded sequence.

Proof. Let $f(z)$ be invertible and let $g(z)$ be its inverse. By hypothesis we have

$$e^{\lambda_n \psi(n)} \, a_n b_n \equiv e^{-\lambda_n \varphi(n)}$$

$$e^{\lambda_n \phi(n)} \, |b_n| = |e^{\lambda_n \phi(n)} \, a_n|^{-1}$$

Since $g(z) = \Sigma b_n \, e^{\lambda_n \varphi(z)} \in F$.We see that $\{|e^{\lambda_n \varphi(z)} a_n|^{-1}\}$ is a bounded sequence

On the other hand, if $\{|e^{\lambda_n \varphi(z)} a_n|^{-1}\}$ bounded sequence, define $g(z)$ such that

$$g(z) = \Sigma \, e^{-2\lambda_n \phi(n)} \, a_n^{-1} \, e^{\lambda_n \psi(z)}$$

Obviously, $g(z) \in F$. Moreover,

$$\begin{aligned} f.g &= \Sigma e^{\lambda_n \varphi(n)} \, a_n \, e^{-2\lambda_n \varphi(n)} \, a_n^{-1} \, e^{\lambda_n \psi(z)} \\ &= \Sigma \, e^{-\lambda_n \varphi(n)} \, e^{\lambda_n \varphi(z)} \\ &= e(z). \end{aligned}$$

Thus,

$$f.g = e.$$

In this section we consider a class G of ETDS such that

$$G = \{f : f(z) = \Sigma a_n \, e^{n \, \varphi(z)} \, , \, \Sigma \, e^{\lambda_n \varphi(z)} \mid a_n \mid < \infty \}$$

The norm in G is defined as

$$\|f\| = \Sigma e^{\lambda_n \varphi(z)} \mid a_n \mid .$$

With similar binary composition as in F and the norm defined in G, G becomes a normed linear space.

Theorem 4. G is a Banach* - Algebra.

Proof. We only prove the completeness.

Let $\{f_p\}$ be a Cauchy sequence in G. Given $\varepsilon > 0$, there exists some $p_0 \geq 1$, such that

$$\|f_p - f_q\| < \varepsilon, \text{ got } p, q, \geq p_0$$

$$e^{\lambda_n \varphi(z)} \, |a_{pn} - a_{qn}| < \varepsilon, \text{ for } p, q \geq p_0. \tag{3.1}$$

This implies that $\{a_{pn}\}$ forms a Cauchy sequence in C for every n and hence, owing to the completeness of C, converges to a complex number, say, a_n. In (3.1), let $q \to \infty$. We get

$$e^{\lambda_n \varphi(n)} \mid a_{pn} - a_n \mid < \varepsilon \, , p \geq p_0$$

$$f_p \to f, \text{ where } f(z) = \Sigma \, a_n \, e^{\lambda_n \varphi(z)} \in G, \text{ since}$$

$$\Sigma e^{\lambda_n \varphi(z)} \, |a_n| \leq e^{\lambda_n \varphi(n)} \, |a_{pn} - a_n| + \Sigma e^{\lambda_n \varphi(n)} \, |a_{pn}| .$$

It can now be verified that G is a Banach algebra.

The proof of the theorem is complete when we define involution mapping.

$$* \, ! \, G \to C$$

as

$$*(f) = f*(z) = \Sigma\, \overline{a}_n\, e^{\lambda_n \psi(z)}$$

Note that G does not becomes a B*-algebra.

Theorem 5. Every continuous linear functional f^* on G is of the form

(3.2) $$f*(f) = \Sigma\, e^{\lambda_n \varphi(z)}\, a_n \,.\, d_n, \text{ where}$$

(3.3) $\{d_n\}$ is a bounded sequence.

Conversely if (3.3) holds, then (3.2) defines a continuous linear functional on G.

Proof. We shall denote that dual space of G by G^*. Let $f^* \in G^*$. Define

$$f_n = e^{\lambda_n \{\psi(z) - \varphi(n)\}} \text{ and } f^N = \sum_{n=1}^{N} a_n\, e^{\lambda_n \psi(z)}$$

obviously, $f^N \to f$ as $N \to \infty$. Let

$$f*(f_n) = d_n.$$

Then

$$f*(f) = f*(\lim_{N\to\infty} f^N)$$

$$= f*\left(\lim_{N\to\infty} \sum_{n=1}^{N} e^{\lambda_n \varphi(z)}\, a_n f_n\right)$$

$$= \sum_{n=1}^{\infty} e^{\lambda_n \varphi(n)}\, a_n \,.\, d_n$$

Moreover,

$$|d_n| = |\, f*(f_n)\,| \le M.\ \|f\| = M.$$

Conversely, let (3.3) holds. The functional defined by (3.2) is evidently well defined and linear. Further we note.

(3.4) $$\|f*\| \le \Sigma e^{\lambda_n \varphi(n)}\, |a_n d_n| \le M\, \|f\|, \qquad \text{by (3.3).}$$

This characterization helps us in formulating an alternative expression for the norm is G^*. We know from (2) that G is a Banach space with the same operations as in G and the norm defined as

$$\|G*\| = \sup_{|f| \le 1} |\, f*(f)\,| / \|f\|.$$

We assert that

$$\|f*\| = \sup_{n \ge 1} |d_n|.$$

By (3.4) we infer that

$$||f^* \,|| = \sup_{\|f\| \leq 1} |f^* (f)| / \|f\| \leq \sup_{n \geq 1} |d_n|.$$

on the other hand

$$|d_n| = |f^* (f_n)| \leq \|f^*\| \cdot \| \; \|f_n\| = \|f^*\|$$

Hence the assertion.

Theorem 6. Let $f(z) = \Sigma\, a_n\, e^{\lambda_n \varphi(z)} \in G, a_n \neq 0,$ for every, $n \geq 1$. Let B be the set of complex numbers having at least one finite limit point. Define

$$f_\alpha (z) = \Sigma a_n\, e^{\lambda_n \{\psi(z) + \psi(\alpha - \varphi(n)\}}.$$

Then the set

$$S_f = (f_\alpha : \alpha \in B)$$

is a total set in G

Proof. Note first that $f_\alpha \in G$, for every $\alpha \in \beta$.

Since

$$f_\alpha(z) = \Sigma e^{\lambda_n \{\psi(z) - \varphi(n)\}}\, a_n \,.\, e^{\lambda_n \psi(z)}, \text{ and}$$

$$\Sigma\, e^{\lambda_n \varphi(n)} \mid a_n\, e^{\lambda_n \{\psi(\alpha) - \varphi(n)\}} \mid = \Sigma \,|a_n|\, e^{\lambda_n \varphi(\alpha)}$$

where $\varphi(\alpha) = \text{Re}\, \psi(\alpha)$,

which must converges for every $\alpha \in C$, since $f(z)$ is a function which converges absolutely in the whole complex plane.

Let $f^* \in G^*$ be such that $f^* (s_f) = 0$ *i.e.*

$$f^* (f_\alpha) = 0, \text{ for every } \alpha \in B$$

implies that $\Sigma\, e^{\lambda_n \psi(n)}\, a_n\, e^{\lambda_n \{\psi(\alpha) - \varphi(n)\}}\, d_n = 0$ for every $\alpha \in B$.

Implies that

(3.5) $\qquad a_n\, e^{\lambda_n \psi(\alpha)}\, d_n = 0$, for every $\alpha \in B$.

Define $h(z) = \Sigma a_n\, e^{\lambda_n \psi(z)}\, d_n$. Since (3.5) holds and $f = \Sigma a_n\, e^{\lambda_n \psi(z)}$ $\in G$, $h = \Sigma a_n\, e^{\lambda_n \psi(z)}\, d_n \in G$. But owing to (3.5)

$$h(\alpha) = 0; \text{ for every } \alpha \in B.$$

Since B has a finite limit point, this means that $h \equiv 0$. This however, implies that $a_n\, d_n = 0$, for every $n \geq 1$, and as a_n is non zero for every n, we get the result.

Theorem 7. Every element of G is a topological zero divisor is G.

Proof. For the definition of topological zero divisor we refer to [1]. Consider the sequence $\{g_k\}$ where

$$g_k(z) = \Sigma\, e^{-\lambda_k \varphi(k)}\; e^{\lambda_k \psi(z)},\; k = 1, 2, 3 \ldots$$

obviously $g_k \in G$ and $\|g_k\| = 1$, and for every $k \geq 1$.

Also,

$$f \cdot g_k = g_k \cdot f = a_k \cdot e^{\lambda_k \psi(z)}$$

So,

$$\| f{\cdot}g_k \| = \| g_k \cdot f \| = e^{\lambda_k \varphi(k)}\, |a_k| \to 0 \text{ as } k \to \infty.$$

Hence, the theorem.

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SOME FIXED POINT THEOREMS IN COMPLETE METRIC SPACES

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ABSTRACT

In this paper we prove fixed point theorems in a complete metric space which extend the theorem 2 of Khan, Swaleh and Sessa [1] and theorem 2 of H.K. Pathak and Rekha Sharma [2].

Introduction

Let R^+ be the set of non-negative real numbers and N the set of positive integers. Khan, Swaleh and Sessa [1] have established fixed point theorems for self maps of complete metric spaces by altering the distance between the points with the use of a function $\psi : R^+ \to R^+$ satisfying the following properties :

(1) ψ is continuous and increasing in R^+;

(2) $\psi(t) = 0$ if and only if $t = 0$.

We denote the set of above function ψ with φ.

In [1, Th.2] the following theorem was proved :

Theorem 1. Let (x, d) be a complete metric space, T a self map of X, and $\psi : R^+ \to R^+$ an increasing, continuous functions satisfying property (2).

Further more, let $a, b, c,$ be three decreasing functions from $R^+/\{0\}$ into $[0, 1]$ such that $a(t) + 2b(t) + c(t) < 1$ for every $t > 0$. Suppose that T satisfies the following condition :

(A) $$\psi(d(Tx, Ty)) \le a(d(x, y))\, \psi(d(x, y)) + b(d(x, y))\, \{\psi(d(x, Tx)) + \psi(d(y, Ty)\} + c(d(x, y)) \min\{\psi(d(x, Ty)), \psi(d(y, Tx))\}$$

where $x, y \in X$ and $x \ne y$. Then T has a unique fixed point.

In (2, Th. 2) the following theorem was proved :

Theorem 2. Let (X, d) be a complete metric space, T a selfmap of X, and $\psi : R^+ \to R^+$ an increasing, continuous function satisfying property (2). Further, let a, b be two decreasing functions from $R^+/\{0\}$ into $[0, 1[$ such that $a(t) + b(t) < 1/2$ for every $t > 0$.

Suppose that T satisfies the following condition :

(B) $$\psi(d(Tx, Ty)) \leq a(d(x, y)) \{\psi(d(x, y)) + c[\psi(d(x, y))\, \psi(d(y, Tx)]^{1/2}\} + b(d(x, y)\ \{\psi(d(x, Tx)) + \psi(d(y, Ty))$$

where $x, y \in X$ and $c \in [0, 1]$ such that $a(t).\,(1 + c) < 1$.

Then T has unique fixed point.

Now we establish the following theorem :

Theorem 3. Let (X, d) be a complete metric space. T is a selfmap of X, and $\psi : R^+ \to R^+$ an increasing continuous function satisfying property (2). Further let a, b be two decreasing functions from$R^+/\{0\}$ into $[0, 1[$ such that $a(t) + b(t) < 1$ for every $t > 0$. Suppose that T satisfies the following condition :

(A)′ $$\psi(d(Tx, Ty) \leq a(d(x, y)) \max \{\psi(d(x, y)), [\psi(d(x, y).\ \psi(d(y, Tx))]^{1/2}\} + b(d(x, y)\ \min\ \{\psi(d(x, Tx), (\psi(d(y, Ty))\}$$

where $x, y, \varepsilon X$. Then T has unique fixed point.

Proof: Let x_0 be a point of X. We define $x_{n+1} = Tx_n, \tau_n = d(x_n, x_{n+1})$ for all $n \in N \cup \{0\}$. We first prove that T has a fixed point. We may assume $\tau_n > 0$ for each n.

From (A)′, we obtain

$$\psi(d(Tx_n, Tx_{n+1}))$$
$$\leq a(d(x_n, x_{n+1})).\ \max.\ \{\psi(d(x_n, x_{n+1})), [\psi(d(x_n, x_{n+1})).\ \psi(d(x_{n+1}, Tx_n))]^{1/2}\} + b(d(x_n, x_{n+1}))\ \min\ \{\psi(d(x_n, x_{n+1})), \psi(d(x_{n+1}, Tx_{n+1}))\}$$

$$\Rightarrow \quad \psi(d(x_{n+1}, x_{n+2})) \leq a(\tau_n).\ \max\ \{\psi\ (\tau_n), [\psi(\tau n).\ \psi(d(x_{n+1}, x_{n+1})]^{1/2}\} + b(\tau_n)\ \min\ (\psi(\tau_n), \psi(\tau_{n+1})\}.$$

If $\psi(\tau_{n+1}) \leq \psi\ (\tau_n)$,. then we obtain

$$\psi(\tau_{n+1}) \leq a(\tau_n).\ \psi(\tau_n) + b(\tau_n).\ \psi(\tau_{n+1}).$$

(3.1) i.e. $$\psi(\tau_{n+1}) \leq \frac{a(\tau_n).}{1 - b(\tau_n)}\ \psi\ (\tau_n) < \psi\ (\tau_n)$$

On the other hand, if $\psi(\tau_n) \leq \psi(\tau_{n+1})$, then we have

(3.2) $$\psi(\tau_{n+1}) \leq [a(\tau_n) + b(\tau_n)]\ \psi(\tau_n) < \psi(\tau_n)$$

Thus $\{\tau_n\}$ is a decreasing sequence since ψ is an increasing.

In (3.1), we put $\lim_{n \to \infty} \tau_n = \tau$ and suppose that $\tau > 0$, then $\tau_n \geq \tau$ implies that

$$\psi(\tau_{n+1}) \leq \frac{a(\tau)}{1 - b(\tau)} \psi(\tau).$$

By letting $n \to \infty$, since ψ is continuous, we have

$$\psi(\tau) \leq \frac{a(\tau)}{1 - b(\tau)} \psi(\tau)$$

a contradiction. So $\tau = 0$.

Similarly if we consider (3.2) then we again have $\tau = 0$.

Now we prove that $\{x_n\}$ is a Cauchy sequence, suppose not, then there exist $\in > 0$ and two sequence $\{p^{(n)}\}, \{q^{(n)}\}$ such that for every

$n \in N \cup \{0\}$, we find that

$p^{(n)} > q^{(n)} \geq n$, $d(x_p^{(n)}, x_q^{(n)}) \geq \in$ and $d(x_{p-1}^{(n)}, x_q^{(n)}) < \in$.

For each $n \geq 0$, we put $S_n = d(x_p^{(n)}, x_q^{(n)})$. Then, we have

$$\in \leq S_n \leq d(x_{p-1}^{(n)}, x_p^{(n)}) + d(x_{p-1}^{(n)}, x_{q-1}^{(n)}) < \tau_{p-1}^{(n)} + \in .$$

Since $\{\tau_n\}$ converges to 0, so that S_n converges to $\in$..

Furthermore, the triangle inequality implies, for each $n \geq 0$,

$$-\tau_p^{(n)} - \tau_q^{(n)} + S_n \leq d(x_p^{(n)} + 1, x_p^{(n)} + 1) \leq \tau_p^{(n)} + \tau_q^{(n)} + S_n,$$

and therefore also the sequence $d(x_{p+1}^{(n)}, x_q^{(n)} + 1)$ converges to $\in$..

Now from (A) we also deduce

$\psi(d(x_{p+1}^{(n)}, x_{q+1}^{(n)}) \leq a(d(x_p^{(n)}, x_q^{(n)})$.

$\max \{\psi(d(x_p^{(n)}, x_q^{(n)}), [\psi(d(x_p^{(n)}, x_q^{(n)}) \,.\, \psi(d(x_q^{(n)}, x_{p+1}^{(n)}))]^{1/2}\}$

$+ b(d(x_p^{(n)}, x_q^{(n)})) \min \{\psi\,(d(x_p^{(n)}, x_{p+1}^{(n)}), \psi(d(x_q^{(n)}, x_{q+1}^{(n)})$

i.e. $\psi(d(x_{p+1}^{(n)}, x_{q+1}^{(n)}) \leq a(s_n) \max \{\psi(s_n), [\psi(s_n),$

$\psi(s_n).\, \psi(d(x_q^{(n)}, x_{p+1}^{(n)}))]^{1/2}\} + b(s_n) \min \{\psi(\tau_p^{(n)}), \psi(\tau_q^{(n)})\}$.

This implies that

$\psi(d(x_{p+1}^{(n)}, x_{q+1}^{(n)}) \leq a(s_n) \max \{\psi(s_n), [\psi(s_n).$

$\psi((\tau_p^{(n)} + s_n)]^{1/2}\} + b\,(s_n) \min \{\psi(\tau_p^{(n)}), \psi(\tau_q^{(n)})\}$.

Letting $n \to \infty$, we are left with

$$\psi(\in) \leq a(\in) \max \{\psi(\in), [\psi(\in).\psi(\in)]^{1/2}\} + b(\in). \min \{\psi(0), \psi(0)\}$$

Therefore, $\psi(\in) \leq a\,(\in)\,.\,\psi(\in) < \psi(\in)$ since $a(t) < 1$ which is absurd. Therefore $\{x_n\}$ is a Cauchy sequence.

By completness of X, $\{x_n\}$ converges to some point z. Now we show that z is a fixed point of T. Since $\tau_n > 0$ there is a subsequence $\{x_n(n)$ of $\{x_n\}$ such that $X_h(n) = z$ for each $n > 0$ and we put $\rho_n = d(z, x_n)$.

Since a, $b < 1$, we obtain from (A')

$$\psi(d(x_{h+1}(n)\,,\,Tz)) = \psi\,.\,(d(T\,x_h(n),\,Tz))$$

$$\leq a(d(Tx_h(n), z))\max\{\psi(d(x_h(n),z)),\ [\psi(d(x_h(n), z)).\ \psi(d(z, x_{h+1}(n)))]^{1/2}\}$$
$$+\ b((d(x_h(n), z)\min\{\psi(d(x_h(n), x_{h+1}(n))),\ \psi\,\{d(z, Tz))\}.$$

$$\Rightarrow \qquad \psi(d(x_{h+1}(n), Tz))$$
$$\leq a(\rho_h\,(n)).\ \max\{\psi(\rho_h\,(n)),\ [\psi(\rho_h\,(n)).\ \psi(d(x_{h+1}(n), z))]^{1/2}\}$$
$$+\ b(\rho_h\,(n))\min\{\psi\,\tau_h(n),\ \psi\,(d(z, Tz))\}$$

Since $\{\rho_n\}$ converges to 0, letting $n \to \infty$, we have

(3.3) $$\lim_{n\to\infty}\ \psi(d(x_{h+1}(n), Tz)) < 0.$$

On the other hand the triangle inequality implies that

$$d(z, Tz) \leq \rho h\,(n) + \tau_h\,(n) + d(x_{h+1}(n)), Tz),$$

which in turn implies that

(3.4) $$\psi(d(z, Tz)) \leq \lim_{n\to\infty}\ \psi(d(x_{h+1}(n), Tz)).$$

Therefore from (3.3) and (3.4), we have

$\psi(d(z, Tz)) = 0$ and therefore $(d(z, Tz)) = 0$.

Then z is a fixed point of T.

If T has two distinct fixed points x, y in X, then

$$\psi(d(x, y) = \psi(d(Tx, Ty))$$
$$\leq a(d(x, y))\max\{\psi(d(x, y)),\ [\psi\,(d(x, y).\ \psi\,(d(y, Tx))]^{1/2}\}$$
$$+\ b(d(x, y))\min\{\psi\,(d(x, Tx)),\ \psi(d(y, Ty))\}$$

$$\Rightarrow \qquad \psi(d(x, y) \leq a(d(x, y))\max\{\psi\,(d(x, y)),\ \psi(d(x, y))\},$$

since $Tx = x$ and $Ty = y$ $\Rightarrow \psi(d(x, y)) < \psi(d(x, y))$ a contradiction. This completes the proof.

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FUNDAMENATAL SOLUTION IN GENERALIZED FUNCTION SPACE IN NEGATIVE RESISTANCE OSCILLATORY CIRCUIT OF DYNATRON OSCILLATOR

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ABSTRACT

Using the method of variation of parameters and a special case of S.M. Joshi Distributional transform, fundamental solutions in generalized function space are utilized to analyze the electronic circuit of Dynatron Oscillator. The appropriate conditions for sustained and decaying oscillations are obtained by the new method.

If the decrease in the applied voltage induced increase in the current, in electronic circuitary arrangement, the resistance offered by this arrangement is called negative resistance. For crimple in a Vacuum tetrode the secondary commission displays the negative resistance. Herold [2] developed the idea (see Fig.1) that the sustained oscillations of high frequency stability can be achieved by shunting a tuned circuit with the negative resistance (here tetrode). The oscillator operators over the negative resistance region *EF* of tetrode plate characteristics. In this region the screen grid voltage becomes much higher than plate voltage for a larger number, than that of primary electrons, of secondary electrons may be produced by the plate and collected by the screen grid. Plate voltages e_{b_1} and e_{b_2} are kept suitably below the screen voltage to offer the negative resistance, due to secondary electron emmission from the plate. The positive resistance of the circuit is neutralized by the negative resistance offered by the tube.

The commercial application of negative resistance is made in dynatron, killitron and transitron oscillators. the dynatron (see Fig. 2), for example, oscillator operates in the negative resistance region of tetrode.

The present paper is concerned with analyzing mathematically the above circuit using the powerful method of fundamental solutions.

Using Kirchoff's law in the equivalent circuit of dynatron oscillator, differential equation for current I_t, after some simplification, takes the form

(1) $\quad [D^2 + \mu D + \nu^2]\, I_t = 0,$ where D stands for the operator d/dt,

$$(2) \qquad \mu = \frac{\gamma_p RC - L}{\gamma_p \,.\, RC} \quad \text{and} \quad \nu^2 = \frac{\gamma_p - R}{\gamma_p \,.\, LC}\,.$$

Although one can use the method of variation of parameters [4] to solve (1) to study the nature of oscillatory current flowing in a dynatron oscillator as has been done in the following section yet that does not at least mathematically, give the whole picture, we therefore obtain the solution with the help of distributional transform

$$(3) \qquad (S^{a_0}_{b_0} f)\,(x) = \langle f(\xi),\, e^{r\,\xi x} \rangle,$$

which is a special case of distributional S.M.Joshi tranform [3]

(4) $(s_b^a f)(x) = < f(\xi), e^{i\xi x} > 0.$

Notation and terminology. Here

(5) $< f, \phi >$

is taken to mean the application of the functional f to the test function ϕ, and $\mathcal{T}$ is the space of basic functions i.e. the space of infinitely differentiable functions which together with all their derivatives approach zero more rapidly than any power of $1/|x|$ as $x \to \infty$.
The generalized function (or the distributions as they are called) Space on T i.e. the set of all linear continuous functionals on $\mathcal{T}$ is denoted by T' as dual space of T.

The linearity of is $\mathcal{T}$ signified by the relation

(6) $< f, \alpha_1\phi_1 + \alpha_2\phi_2 > = \alpha_1 < f, \phi_1 > + \alpha_2 < f, \phi_2 >$

where as the continuity of the generalized function f implies that, if the sequence $\phi_1, \phi_2, ..., \phi_\nu, ...$ converges to zero in $\mathcal{T}$ then the sequence $< f, \phi_1 >, < f, \phi_2 >$, ... also converges to zero in T'. For regular functionals

(7) $< f, \phi > = \int f(x)\, \phi(x)\, dx.$

Functionals that can not be represented as (7), are called *singular.*

For example the delta functional represented by

(8) $< \delta(x), \phi(x) > = \phi(0)$

is singular and so is the so called "translated" [1] delta functional is defined by

(9) $< \delta(x - x_0), \phi(x) > = \phi(x_0).$

Sustained oscillations

To obtain solution of (1) by the method of fundmental solution let it has the driving impulse $-g(t, t')$ originating from a source point at the instant t'. We now concentrate on the solution of the differential equation

(10) $[D^2 + \mu D + \gamma^2]\, I_t = -\, g(t, t').$

The wellknown variation of parameter method then gives the solution as

(11) $I_t = [A(t) \cos \rho t + B(t) \sin \rho t] \exp(\sigma t),$ if $4\nu^2 > \mu^2$

and

(12) $I_t = [A(t) \cosh \rho\, t + B(t) \sinh \rho t] \exp(\sigma t),$ if $4\nu^2 < \mu^2.$

where

(13) $\rho^2 = \frac{1}{2}(\mu^2 - 4\nu^2)$ and $\sigma = -\, \frac{1}{2}\mu$

The variation of parameter method then requires, to obtain the parameters, that

(14) $$[\rho\{\frac{dB(t)}{dt} - (\tfrac{1}{2}\mu^2 - \nu^2)A(t)\} \cos \rho t$$
$$-\, \{\rho\, \frac{dA(t)}{dt} + (\tfrac{1}{2}\mu^2 - \nu^2)B(t)\} \sin \rho t] \exp(\sigma\, t) = -\, g(t, t')$$

Solution could then be obtained from (11) and (14). But in practice it is difficult to be obtained unless we take

(15) $\mu^2 = 2\nu^2$ or in other words

(16) $$\left[\frac{\gamma_p RC - L}{\gamma_p LC}\right]^2 = 2\left[\frac{\gamma_p - R}{\gamma_p LC}\right].$$

This is exactly the condition of oscillations in Dynatron oscillators.

This gives $\gamma_p = \frac{1}{2C}\left(1 - \frac{R^2}{L}\right)^{-1}$, or $= \frac{1}{2C}\left(\frac{R^2}{L} - 1\right)^{-1}$.

Under the assumption of negative plate resistance, which as explained earlier in the fundamental requirement in the Dynatron oscillator, only the later value is edmissible. Hence

(17) $w_0^2 R^2 C \leq 1,$

where as usual w_0 is the resonant frequency.

The dumping factor μDI_t helps to decay the oscillations.

The sustained oscillations, then, can only be obtained if

(18) $\mu = 0$ and $\gamma_p = L/CR$.

The frequency of oscillation is, then, given by

(19) $$f = \frac{LC}{2\pi} - \frac{1}{2}\left(1 - \frac{R}{\gamma_p}\right)^{1/2}.$$

Equation (1) under these conditions is that of simple Harmonic oscillator. However practically such an adjustment is difficult to obtain.

Fundamental Solution in τ'.

A fundamental solution in τ' is the solution of the equation

(20) $[D^2 + \mu D + \nu^2]\, F(t, t') = -\,\delta(t, t'),$

where $\delta(t, t')$ is the "translated" generalized delta function defined by [1], of unit impulse.

We take the timings t and t' when the oscillator operates at points E and F of Fig.2 respectively.

Once $F(t, t')$ is known the solution for oscillatory current equation (10) can be obtained [1] as

(21) $I_t = F(t) * g(t),$

where * denotes convolution operation in T'.

But taking Distributional transform (4) of both sides of (20) one obtains

$$F(t, t') = \frac{1}{\xi^2 + \mu\xi - \nu^2}, \qquad \text{where } F(t) = (s^{a_0}_{b_0} F).$$

Taking inverse [3] we easily obtain

(22) $F(t, t') = \frac{1}{2\pi} < (\rho^2 + i\mu\xi - \nu^2)^{-1}, exp\ t\ \xi(t - t')>$

Hence

(23) $F(t, t') = \{2(4\nu^2 - \mu^2)^{-½} \sin[(\nu^2 - \mu^2/4)^{1/2}(t - t')]\ expt\ (t - t')^{½2}\}$

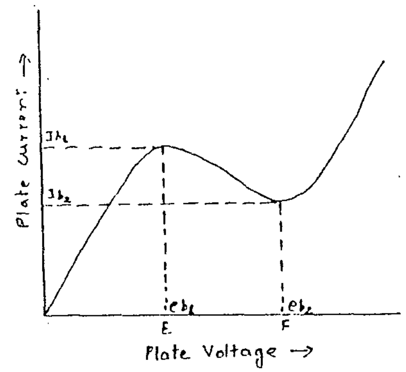
$$\{H(t-t')\},$$

where $H(t-t')$ is the Heavisides unit step Function.

Discussion

To obtain sustained oscillations it is necessary that γ_p be given as closely as given by (18) so that harmonically oscillating system of oscillating frequency 'f' (given by (19) may be function.

This is verified by the forms of solution (23) which shows that the oscillation dies out exponentially resulting in failure of dynatron oscillator.



(FIG 1) TETRODE CHARACTERISTIC

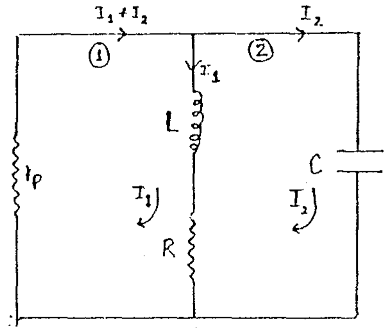


FIG 2(b) - EQUIVALENT CIRCUIT OF DYNATRON.

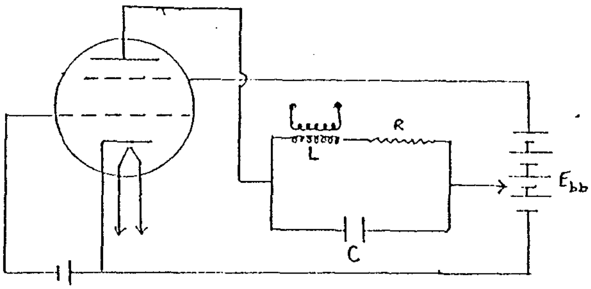


FIG 2(a) DYNATRON OSCILLATOR

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SCATTERING IN GENERALIZED FUNCTION SPACE

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1. INTRODUCTION

The most direct information about the nature of forces between the particles is obtained from the study of collisions. The experimental data can be interpreted in terms of the model of the miscroscopic details of a single collision and the conclusions concerning the interaction between the scattered particle and the scatter can be drawn. In recent years this study has assumed quite importance because of its wide applications.

Attempts have been made [1, 4, 5] to analyse this type of scattering by several authors but all of them have givn aproximate methods and adoption of these analytical theories makes it difficult to choose the appropriate value of parametres thereby restricting the proper choice of energy values of the system and the symmetry of the potential function.

The present paper attempts to solve the corresponding differential equation in a general approximation free way in the so called generalized function space (or distribution space). The technique adopted substantially expands the range of problems that can be tackled.

In what follows it is assumed that the interaction between the scattered particle and the scatterer can be represented by the potential energy function $V(\boldsymbol{r})$, where $\boldsymbol{r}$ is the vector joining the scattered particle and the centre of force. At present we restrict ourselves to elastic scattering only in which the kinetic energy of the system is not changed due to the collision. If the mass of the scatterer is large as compared to that of scattered particle e.g. in the collision between an electron and an atom the scatterer can be assumed to remain at rest during the entire collision process.

2. Notation and Terminology

A generalized function is a generalization of the classical notion of a function. It reflects the fact that in reality one can not measure the value of a physical quantity at a point, but can only measure the mean values within sufficiently small neighbourhoods of the point and then proclaim the limit of the sequence of those mean values as the value of the physical quantity at the given point. Thus it enables one to express in regorous mathematical form such ideal concepts as density of a point charge or dipole and the Paul Dirac's delta function.

We shall be interested in finding the solution of the differential equation of the scattering process discussed in the last para of Section 1, viz.

(2.1) $\quad (\nabla^2 + k^2) \;=\; \dfrac{2m}{h^2}\, V(\boldsymbol{r}),$

where $k^2 = \dfrac{2m}{h^2} E$

in the space τ[2] of basic functions i.e. the space of infinitely differentiable functions which, together with their derivatives, aproach zero more rapidly than any power of $1/|x|$ as $|x| \to \infty$. The generalized function space of τ i.e. the set of continuous linear functionals on τ shall be denoted by τ' as dual space of τ. The space τ' is known as the space of generalized functions of slow growth and is very important in applications of Mathematical Physics. The terminology used here is that of [2].

3. Formulation of the problem

In general, the potential energy $V(\boldsymbol{r})$ decreases in magni+ude as the distance $r = |\hat{\boldsymbol{r}}|$ from the scattering centre becomes large, it is convenient to choose the arbitrary constant in the definition of $V(\boldsymbol{r})$ such that $V = 0$ at $r = \infty$.

If $V(r)$ decreases to zero sufficiently rapidly as $r \to \infty$, the particle can be considered essentially free when r is large. The asymptotic wave function is therefore, in general, a linear superposition of the free particle wave function *exp* $(i\boldsymbol{k}.\boldsymbol{r})$.

A solution of (2.1) is sought which satisfies the boundary condition

(3.1) $\quad \psi \sim \exp\,(i\,\hat{\boldsymbol{k}}.\,\hat{\boldsymbol{r}}) + f(k, k')\, eikr/r$

,where r is large enough for the particle to be beyond the range of force. The scattering amplitude is $f(k, k')$, where k and k' are propogation vectors.

It is convenient to separate the incidental wae function *exp* $(i\,\boldsymbol{k}.\,\boldsymbol{r})$ from the wave function and to write

(3.2) $\quad \psi = \exp\,(i\,\hat{\boldsymbol{k}}\,.\,\hat{\boldsymbol{r}}) + \psi_s$

where ψ_s the scatterer wave, is asymptotic to e^{ikr}/r. Since $\exp\,(i\,\hat{\boldsymbol{k}}\,.\,\hat{\boldsymbol{r}})$ is a solution of homogeneous wave equation

(3.3) $\quad (\nabla^2 + k^2)\exp\,(i\,\hat{k}\,.\,\hat{r}) = 0,$

the corresponding Schrodinger's equation takes the inhomogeneous form

(3.4) $(\nabla^2 + k^2)\,\psi_s = \dfrac{2m}{h^2}\,V(r)\psi.$

A formal solution of (3.4) can now be obtained in generalized function space τ'.

4. The solution in the space τ' .

Let us take

(4.1) $\dfrac{2m}{h^2} - V(r)\psi = -\,\rho(r)$

Here the quantity $\rho(r)$ can be regarded as a source density function for scattered divergent spherical waves.

Before finding the actual solution it may be pointed out that if $\psi(s_1)$ and $\psi(s_2)$are solutions of (3.7), with $\rho(r)$ as in (4.1), belonging to density functions $\rho_1(r)$ and $\rho_2(r)$ satisfying

(4.2) $< \psi s_1, \phi > < f_1, e^{ikr}/r, \ \phi>$, and

(4.3) $< \psi_2\, \phi > \sim < f_2\, e^{ikr}/r, \phi>$

respectively; then $\psi_s = \psi_{s_1} + \psi_{s_2}$ is a soluiton belonging to $f(r) = \rho_1(r) + \rho_2(r)$ such that

(4.4) $< \psi_S, \phi > \sim < f, e^{ikr}/r, \phi>,$

where $f = f_1 + f_2$ is a well defined linear continuous functional defined over the space of density functions $\rho_1(r)$ and $\rho_2(r)$.

Here

(4.5) $< f, \phi >$

is taken to mean as the application of the functional $f \in \tau'$ to the test function $\phi \in \tau$.

Also the linearity of f is signified by the relation

(4.6) $< f, \alpha_1\phi_1 + \alpha_2\phi_2> = < \alpha_1 ; f, \phi_1 > + < \alpha_2\ f, \phi_2>$

where as continuity of the generalized function f implies that if the sequence $\phi(, \phi_2, \ldots, \phi_v, \ldots$ converges to zero in τ then the sequence $< f, \phi_1>, <f, \phi_2> , \ldots, <f_1, \phi_v>, \ldots$ also converges to zero.

For *regular* functionals

4.7 $< f, \phi> = \int f(x)\, \phi\, (x)\, dx.$

Functionals that can not be represented as (4.7) are called *singular*.

For example the delta functional represented by

(4.8) $< \delta(x), \phi(x) > = \phi(0)$

is singular, and so is the socalled "translated" delta functional defined by

(4.9) $\quad < \delta\,(x - x_0), \phi(x)> = \phi(x_0).$

Now to obtain the desired solution let $E(r, r')$ be the solution of the equation

(4.10) $\quad (\nabla^2 + k^2)\, E(r, r') = -\,\delta(r, r'),$

where $E(r, r')$ is asymptotic to e^{ikr}/r.

Here $\delta(r, r')$ is the source density of unit strength.

Using the distributional Fourier transform technique, we have

(4.11) $\quad E(r, r') = \dfrac{1}{2\pi} < E(r, r'), e^{-\,ik(r - r')\xi}>,$

where $E(r, r')$ denotes the distributional Fourier transform of

$E(r, r')$ *i.e.* $E(r, r') = 1/k^2 - \xi^2$, using (4.10).

The solution for the scattering problem for the density function $\rho(r)$ can then be calculated from the equation

(4.12) $\quad \psi_s = \rho(r)^*\, E(r, r').$

Here * denotes the convolution defined by

(4.13) $\quad < f * g, \phi > = < f(x) \times g(y), \phi\,(x + y)> .$

For example

$$< \delta * f, \phi > = < \delta(x) \times f(y), \phi(x + y) >$$
$$= < f(y), < \delta\,(x), \phi(x + y)>> = < f(y), \phi(y)> = < f, \phi> .$$

Thus for any generalized function f and the generalized delta function defined by (4.9), we have

(4.14) $\delta * f = f * \delta = f$

and that, in general,

(4.15) $\quad f * g = g * f$

holds also for generalized functions f and g in which the convolution operation has a meaning.

Now, using (4.9), the arbitrary density $\rho(r)$ is given by

(4.16) $\quad \rho(r) = < \delta(r, r'),\ \rho(r')>,$

where $\rho(r')$ is the density function of point source at r'.

The scattering problem is thus formulated in the form of an integral equation, by (4.1) and (3.2)

(4.17) $\quad \psi(r) = \exp\,(i\,\hat{\boldsymbol{k}}\,.\,\hat{\boldsymbol{r}}) - \dfrac{2m}{h^2} < E(r, \boldsymbol{r}'), V(r')\ \ \psi\,(r')>$

5. Discussion

Equation (4.17) shows that the scattered wave at the point r is composed of spherical waves $E(r, r')$ originating at each point of space

r. The amplitude of each contribution is proportional to the product $V(r')\,\psi(r')$ i.e. it is porportional jointly to the strength of the interaction and the amplitude of wave function at r'. All these spherical waves are compounded at the point r and generate the total scattered wave, which is then added to the incident wave to produce the total wave function at r.

Also, if the potential energy function is assumed to be confined to a limited region of space, the asymptotic form of $E(r, r')$ as indicated at (4.10) can be substituted in (4.17) to yield the scattering amplitude

$$(4.18)\ f(k, k') = -\frac{2m}{h^2} < \exp(-\boldsymbol{ik} \cdot \boldsymbol{r}'), V(r')\,\psi(r')>$$

Equation (4.18) is a general formula by which the scattering cross section $|f(k, k')|^2$ can be computed.

Conclusion

The solution thus achieved are more general in nature than those found in the existing literature as they are neither subjected to any restriction on the energy values of the system nor on the symmetry the potential function. More over the results may find applications in the study of matter in bulk. Although, (4.17) does not provide an explicit solution as appears inside, yet the formulation of the problem according to (4.17) provides a single equation which expresses both the content of the differential equation (2.1) and the boundry condition (3.1).

Also the method applies equally suitably, if instead of the distributional Fourier transform one takes its wider generalization (introduced by the first author recently in 3), called S.M. Joshi generalized Fourier transform, given by

$$(5.1)\qquad (S_b^a f)(x) = \frac{\Gamma(a)\,\Gamma(b-\alpha)}{\Gamma(b)\Gamma(a-\alpha)} < f(\xi),\ {}_1F_1(a;b;i\,\xi\,x)>,$$

where $a = \alpha + \lambda + 1$, $b = a + \mu$, and ${}_1F_1$ is the confluent hypergeometric function. For $\mu = 0 = \alpha$, (5.1) reduces to the distributional Fourier transform

$$(5.2)\qquad (S_{b_0}^{a_0} f)(x) = <f, e^{i\xi x}>.$$

Last, but the most fruitful, advantage of the method is the fact that it applies to both regular (the classical one) and singular functional, as pointed out in (4.7) and the line following it, and it utilizes the most modern concept of generalized delta function, which existed in the literature previously only in mathematically non rigorous form-a form about which its originator Paul Dirac (See "The Principles of Quantum Mechanics, "Clarendon Press, Oxford (1958) by Dirac) himself had doubts that whether it was a function in the classical meaning.

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INTEGRATION OF I-FUNCTION OF TWO VARIABLES WITH RESPECT TO PARAMETERS

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ABSTRACT

The object of the present paper is to establish four integrals associated with I-function of two variables due to Goyal and Agrawal [3]. The integration is performed with respect to a parameter. Such integrals are usuful in the study of certain boundary value problems.

1. INTRODUCTION

The I-function of two variables represented by means of two contour integrals is defined by Goyal and Agarwal [3] and represented in as following manner :

$$I\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \frac{1}{(2\pi w)^2} \int_{l_1} \int_{l_2} \Phi_1(\xi)\, \Phi_2(\eta)\, \Psi(\xi, \eta)\, z_1^{\xi} z_2^{\eta}\, d\xi\, d\eta$$

where

$$w = \sqrt{-1},$$

$$\Phi_1(\xi) = \frac{\prod_{j=1}^{m_2} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{n_2} (-a_j + \alpha_j \xi)}{\sum_{i=1}^{r} \left[\prod_{j=m_2+1}^{q_i^{(1)}} \Gamma(1 - b_{ji} + \beta_{ji} \xi) \prod_{j=n_2+1}^{p_i^{(1)}} \Gamma(a_{ji} - \alpha_{ji} \xi) \right]} \quad \ldots (1.2)$$

$$\Phi_2(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(d_j - \delta_j \eta) \prod_{j=1}^{n_3} \Gamma(1 - c_j + \gamma_j \eta)}{\sum_{i=1}^{r} \left[\prod_{j=m_3+1}^{q_i^{(2)}} \Gamma(1 - d_{ji} + \delta_{ji} \eta) \prod_{j=n_3+1}^{p_i^{(2)}} \Gamma(c_{ji} - \gamma_{ji} \eta) \right]} \quad \ldots (1.3)$$

$$\Psi(\xi, \eta) = \frac{\prod_{j=1}^{m_1} \Gamma(f_j - F_j(\xi+\eta)) \prod_{j=1}^{n_1} \Gamma(1 - e_j + E_j(\xi+\eta))}{\prod_{j=m_1+1}^{q} \Gamma 1 - f_j + F_j(\xi+\eta \prod_{j=n_1+1}^{p} \Gamma e_j - E_j(\xi+\eta)} \quad \ldots (1.4)$$

z_1 and z_2 both are not equal to zero.

$$A = \sum_{j=1}^{n_1} E_j - \sum_{j=n_1+1}^{p} E_j + \sum_{j=1}^{m_1} F_j - \sum_{j=m_1+1}^{q} F_j + \sum_{j=1}^{m_2} \beta_j - \sum_{j=m_2+1}^{q_i^{(1)}} B_{ji}$$

$$+ \sum_{j=1}^{n_2} \alpha_j - \sum_{j=n_2+1}^{p_i^{(1)}} \alpha_{ji} > 0 \quad \ldots (1.5)$$

$$B = \sum_{j=1}^{n_1} E_j - \sum_{j=n_1+1}^{p} E_j + \sum_{j=1}^{m_1} F_j - \sum_{j=m_1+1}^{q} F_j + \sum_{j=1}^{m_3} \delta_j - \sum_{j=m_3+1}^{q_i^{(2)}} \delta_{ji}$$

$$+ \sum_{j=1}^{n_3} \gamma_j - \sum_{j=n_3+1}^{p_i^{(2)}} \gamma_{ji} > 0 \quad \ldots (1.6)$$

$$\forall\ i = 1, 2, \ldots r$$

$$|arg\, z_1| < \frac{A\pi}{2}, \quad |arg\, z_2| < \frac{B\pi}{2}.$$

For asymptotic expansion and all other conditions, see Goyal Anil and Agrawal R.D. [3]. These conditions will be assumed to hold good in the I-function of two variables occuring in this paper.

2. FORMULAE REQUIRED

In this paper, we require the following results

It is a hypergeometric function with unit argument given by Whittakar and Watson [2]

$$\frac{1}{2\pi w} \int_{-i\infty}^{+i\infty} \frac{\Gamma(a+x)\,\Gamma(b-x)\,\Gamma c - x)}{\Gamma(d-x)}\, e^{\pm i\pi x}\, dx$$

$$= \frac{\Gamma(a+b)\,\Gamma(a+c)\,\Gamma(d-a-b-c)}{\Gamma(d-b)\,\Gamma(d-c)}\, e^{\pm i\pi a}$$

$$\mathrm{Re}(d-a-b-c) > 0 \quad \ldots (2.1)$$

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1), & n \geq 1 \\ 1 \quad \text{for} & n = 0,\ \alpha \neq 0 \end{cases} \quad \ldots (2.2)$$

3. INTEGRALS

The integrals involving the I-function of two variables to be evaluated are

(I) $$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma b-x\,\Gamma c-x}{\Gamma d-x}\,e^{\pm i\pi x}\; I^{m_1,n_1+1:P}_{p+1,q+1:Q}\left[\begin{matrix}z_1\\ z_2\end{matrix}\left|\begin{matrix}(1-a-x:h,k),(e_p,E_p):T\\ (f_q,F_q)\qquad\qquad :T^1\end{matrix}\right.\right]dx$$

$$=\frac{e^{\pm i\pi a}}{\Gamma\, d-b\,\Gamma\, d-c}\, I^{m_1+1,n_1+2:P}_{p+2,q+1:Q}\left[\begin{matrix}z_1e^{\pm i\pi h}\\ z_2^{\pm i\pi k}\end{matrix}\left|\begin{matrix}(1-a-b:h,k),(1-a-c:h,k)\,(e_p,E_p):T\\ (d-a-b-c:h,k),(f_q,F_q)\qquad T^1\end{matrix}\right.\right]dx \quad \text{... (3.1)}$$

provided $h, k > 0$

$$\mathrm{Re}\left[d-a-b-c+h\min_{1\le j\le m_2} R\left(\frac{b_i}{\beta_j}\right)+k\min_{1\le j\le m_3} R\left(\frac{d_j}{\delta_j}\right)\right]>0 \quad \text{... (3.1.1)}$$

$$P=m_2,n_2;m_3,n_3,\quad Q=p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:r \quad \text{... (3.12)}$$

$$T=[(a_j,\alpha_j)_{1,n_2}],[(a_{ji},\alpha_{ji})_{n_{2+1}},p_i^{(1)}];[c_j,\gamma_j)_{1,n_3},[(c_{ij},\gamma_{ji})_{m_3+1},p_i^{(2)}] \quad \text{... (3.13)}$$

$$T^1=[(b_j,\beta_j)_{1,m_2},[(b_{ji},\beta_{ij})_{m_2+1,q_i^{(1)}}];[(d_j,\delta_j)_{1,m_3}],[(d_{ji},\delta_{ji})_{m_3+1,q_i^{(2)}}] \quad \text{... (3.14)}$$

(II) $$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\Gamma(a+x)\,\Gamma(b-x)\,\Gamma\, c-x\, e^{\pm i\pi x}$$

$$I^{m_1,n_1:P}_{p+1,q:Q}\left[\begin{matrix}z_1\\ z_2\end{matrix}\left|\begin{matrix}(e_p,E_p),(d-x:h,k):T\\ (f_q,F_q)\qquad\qquad :T^1\end{matrix}\right.\right]dx$$

$$=e^{\pm i\pi a}\Gamma(a+b)\,\Gamma\,(a+c)\, I^{m_1+1,n_1:P}_{p+2,q+1:Q}\left[\begin{matrix}z_1\\ z_2\end{matrix}\left|\begin{matrix}(e_p,E_p),(d-b:h,k),(d-c:h,k):T\\ (d-a-b-c:h,k),(f_q,F_q))\qquad :T^1\end{matrix}\right.\right] \quad \text{... (3.2)}$$

provided $h, k > 0$

$$\operatorname{Re}\left[d-a-b-c-h \min_{1\le j\le m_2} \operatorname{Re}\left(\frac{b_i}{\beta_j}\right)+k \min_{1\le j\le m_3} \operatorname{Re}\left(\frac{d_j}{\delta_j}\right)\right]>0 \qquad \ldots (3.2.1)$$

(III) $$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\Gamma(a+x)\,\Gamma(b-x)\,\Gamma(c-x)\,e^{\pm i\pi x}$$

$$I^{m_1, n_1+1:P}_{p+2, q:Q}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\left|\begin{matrix}(1-c+x:h,k),(e_p,E_p),(d-x;h,k):T \\ (f_q,F_q) \qquad :T^1\end{matrix}\right.\right]dx$$

$$=e^{\pm i\pi a}\Gamma(a+b)\,I^{m_1+n_1+1:P}_{p+3,q+1:Q}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\right.$$

$$\left|\begin{matrix}(1-a-c:h,k),(e_p,E_p),(d-b:h,k),(d-c:2h,2k):T \\ (d-a-b-c:2h,2k),(f_q,F_q)) \qquad :T^1\end{matrix}\right] \qquad \ldots (3.3)$$

provided $h, k > 0$

$$\operatorname{Re}\left[d-a-b-c-2h \min_{1\le j\le m_2} R\left(\frac{b_j}{\beta_j}\right)-2k \min_{1\le j\le m_3} R\left(\frac{d_j}{\delta_j}\right)\right]>0 \qquad \ldots (3.3.1)$$

(IV) $$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma(a+x)\,\Gamma(b-x)}{\Gamma(c-x)}\,e^{\pm i\pi x}\; I^{m_1, n_1+1:P}_{p+1, \; q \; :Q}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\right.$$

$$\left|\begin{matrix}(1-d-x:h,k),(e_p,E_p):T \\ (f_q,F_q) \qquad :T^1\end{matrix}\right]dx$$

$$=e^{\pm i\pi a}\frac{\Gamma(a+b)}{\Gamma(c-b)}\,I^{m_1+1, n_1+1:P}_{p+2, q+1:Q}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\right.$$

$$\left|\begin{matrix}(1-a-d):h,k),(e_p,E_p),(c-d:h,h,k):T \\ (c-a-b-d:h,k),(f_q,F_q) \qquad T^1\end{matrix}\right]dx \qquad \ldots (3.4)$$

provided $h, k > 0$

$$\operatorname{Re}\left[c-a-b-d-h \min_{1\le j\le n_2} R\left(\frac{a_j-1}{\alpha_j}\right)-k \max_{1\le j\le n_3} R\left(\frac{c_j-1}{\gamma_j}\right)\right]>0 \qquad \ldots (3.4.1)$$

Here P, Q, T, T^1 are defined as per equations (3.1.2), (3.1.3), (3.1.4) respectively.

Proof. To prove the integral (3.1), we replace I-function of two variables by (1.1), change the order of integration which is justified under the conditions stated in (1.5), (1.6). We get

$$\frac{1}{(2\pi w)^2}\int_{i_1}\int_{i_2} \Phi_1(\xi)\,\Phi_2(\eta)\,\psi(\xi, n)\, z_1^{\xi} z_2^{\eta}$$

$$=\left\{\frac{1}{2\pi i}\int_{-ix}^{+i\infty} \frac{\Gamma b-x\,\Gamma a-x+h\xi+k\eta\,\Gamma c-x}{\Gamma d-x} e^{\pm i\pi x}\,dx\right\} d\xi\, d\eta \quad \ldots (3.5)$$

Now we evaluate the inner integral using a known integral formula given by Whittakar and Watson [2], noting that it is a hypergeometric function with unit argument.

On expressing the resulting expression with the help of (1.1) we obtain (3.1).

Proceeding similarly we easily evaluate the integrals (3.2), (3.3), (3.4) respectively.

PARTICULAR CASES

(i) If we take $r = 1$, all greek letters equal to unity in (1.1), the I-function of two variables would reduce to relatively more familiar G-function of two variables. Thus our result (3.1), (3.2), (3.3), (3.4) will yield similar integrals involving the G-function of two variables.

(ii) If we set $r = 1, m_1 = 0$ in (3.1), (3.2), (3.3), (3.4), we get integrals for the H-function of two variables.

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COMMON FIXED POINT OF TWO PAIRS OF COMMUTING MAPPING IN SAKES SPACE

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ABSTRACT

Meir and Keeler [1] obtained a remarkable generalisation of Banach contraction principle. In this paper, a common fixed point theorem for the pairs of commuting mapping in a Saks space, satisfying Meir and Keeler type condition is obtained.

Theorem Let P and S be commuting mappings and Q and T be commuting mappings of a Saks space $(Xs.d) = (X.N_1 . N_2)$ into itself satisfying the following conditions :

Given $\in > 0$, there exists a $\delta(\in) > 0$, ($\delta(\in)$ being a non decreasing function of $\in$) such that for all $x, y \in X$, $\in \leq K \max \{N_2(Sx - Ty),$ $N_2(Px - Sx), N_2(Qy - Ty), N_2(Px - Ty),\quad N_2(Sx - Qy)\} < \in + \delta$

$$\Rightarrow N_2(Px - Qy) < \in \qquad \text{... (1)}$$

$$Px = Qy \text{ whenever } Px = Sx,\ Qy = Ty \qquad \text{... (2)}$$

where $0 \leq 2K < 1$.

Further if the range of T contains the range of P and the range of S contains the range of Q and if one of P, Q, S and T is continuous then P, Q, S and T have a common fixed point z. Further z is the unique common fixed point of P and S of Q and T.

Proof : For the proof of this theorem we recall the lemma of Orlicz [2] for a Saks space.

Lemma Let $(Xs, d) = (X, N_1, N_2)$ be a Saks space. Then the following statements are equivalent.

(*i*) N_1 is equivalent to N_2 on X

(*ii*) (X, N_1) is a Banach space and $N_1 \leq N_2$ on X

(*iii*) (X, N_2) is a Frechet space and $N_2 \leq N_1$ on X.

Now we shall prove the theorem. First with the help of (1), we note that for all x, y in X such that

$$Px \neq Sx,\ Qy \neq Ty,\ N_2(Px - Qy) < K \max \{N_2(Sx - Ty), N_2(Px - Sx)$$
$$N_2(Qy - Ty), N_2(Px - Ty), N_2(Sx - Qy)\}.$$

Next the non decreasing character of $\delta(\in)$ implies that given $\in > 0$, there exists $\in > 0$ such that

$$\in' < \in < \in' + \delta(\in') \text{ or equivalently}$$
$$K \max \{N_2(Sx - Ty), N_2(Px - Sx), N_2(Qy - Ty), N_2(Px - Ty), N_2(Sx - Qy)\} = \in \quad \Rightarrow N_2(Px - Qy) < \in', \in' < \in \quad \ldots (4)$$

Let x_0 be an arbitrary point in X Choose a point x_1 and then a point x_2 in X such that $Px_0 = Tx_1$ and $Qx_1 = Sx_2$.

This can be done as $PX \subset TX$. In general having chosen the point x_{2n} choose a point x_{2n+1} and then a point x_{2n+2} such that

$$Px_{2n} = Tx_{2n+1} \text{ and } Qx_{2n+1} = Sx_{2n+2}.$$

Then we have,

$$N_2(Px_{2n} - Qx_{2n+1}) < K \max \{N_2(Sx_{2n} - Tx_{2n+1}), N_2(Px_{2n} - Sx_{2n}), N_2(Qx_{2n+1} - Tx_{2n+1}), N_2(Px_{2n} - Tx_{2n+1}, N_2(Sx_{2n} - Qx_{2n+1})\},.$$

$$N_2(Px_{2n} - Qx_{2n+1}) < K \max \{N_2(Qx_{2n-1} - Px_{2n}), N_2(Px_{2n} - Qx_{2n-1}), N_2(Qx_{2n+1} - Px_{2n}\}, N_2(Px_{2n} - Px_{2n}), N_2(Sx_{2n} - Qx_{2n+1})\}$$

$$N_2(Px_{2n} - Qx_{2n+1}) < K \max \{N_2(Qx_{2n-1} - Px_{2n}), N_2(Qx_{2n-1} - Qx_{2n+1})\}$$
$$= K \max \{N_2(Qx_{2n-1} - Px_{2n}), N_2(Qx_{2n-1} - Px_{2n}) + N_2(Px_{2n} - Qx_{2n+1})\}.$$

Or $N_2(Px_{2n} - Qx_{2n+1}) < K[N_2(Qx_{2n-1} - Px_{2n}) + N_2(Px_{2n} - Qx_{2n+1}\}$,

i.e. $\quad N_2(Px_{2n} - Qx_{2n+1}) < K/(1-K)N_2(Qx_{2n-1} - Px_{2n})$.

Similarly we have

$N_2(Qx_{2n-1} - Px_{2n}) = N_2(Px_{2n} - Qx_{2n-1}) < K/(1-K)N_2\,(Qx_{2n-1} - Px_{2n-2})$.

From the last two inequalities we conclude that both $N_2(Px_{2n} - Qx_{2n+1})$ and $N_2(Qx_{2n+1} - Px_{2n+2}) = 0$ are monotonic decreasing sequences of positive real numbers.

Now $\lim\limits_{n \to \infty} N_2(Px_{2n} - Qx_{2n+1}) = 0$

and $\lim\limits_{n \to \infty} N_2(Qx_{2n+1} - Px_{2n+2}) = 0$.

For, if not, suppose for instance

$\lim\limits_{n \to \infty} N_2(Px_{2n} - Qx_{2n+1}) = r$ where $r > 0$.

Then given $\delta > 0$ there exists a positive integer N such that for each integer $m > N$, we heave

$$r \le N_2(Px_{2m} - Qx_{2m+1}) < r + \delta \quad \ldots (5)$$

Or $r \le K \max \{N_2(Sx_{2m+2} - Tx_{2m+1}), N_2(Px_{2m+2} - Sx_{2m+2}),$

$N_2(Qx_{2m+1}) - Tx_{2m+1}),$
$$N_2(Px_{2m+2} - Tx_{2m+1}), N_2(Sx_{2m} - Qx_{2m+1})\} < r + \delta \quad \text{... (6)}$$

Selecting δ in (6) in accordance with (1), for each $m \geq N$ we obtain $N_2(Px_{2m} - Qx_{2m+1}) < r$ and so, $N_2(Px_{2m+2} - Qx_{2m+3}) < r$ which contradicts (5). Therefore we have

$$\lim_{n \to \infty} N_2(Px_{2n} - Qx_{2n+1}) = 0$$

and similarly, $\lim\limits_{n \to \infty} N_2(Qx_{2n+1} - Px_{2n+2}) = 0.$

Now by lemma,

$\lim\limits_{n \to \infty} N_1(Px_{2n} - Qx_{2n+1}) = 0 = \lim\limits_{n \to \infty} N_1(Qx_{2n+1} - Px_{2n+2})$.From these limits and by condition (1) it follows easily that $\{Px_0, Qx_1, Px_2, Qx_3, ..., Px_{2n}, Qx_{2n+1}, ...)$ is a Cauchy sequence in the Saks space X and so has a limit z in X.

Hence the sub-sequence $\{Px_{2n}\} = \{Tx_{2n+1})$ and $\{Qx_{2n+1})$ $= \{Sx_{2n+2}\}$ converge to the point z.

Let us now suppose that the mapping S is continuous. Then since the mappings P and S commute, the sequences $\{Sx_{2n}\}$ and $\{PSx_{2n}\}$ converge to the point Sz. Now we shall show that $Sz = z$. For if $z \neq Sz$, then $N_2(PSx_{2n} - Qx_{2n+1} < K \max\{S^2x_{2n} - Tx_{2n+1}),$
$N_2(PSx_{2n} - S^2x_{2n}), N_2(Qx_{2n+1} - Tx_{2n+1}), N_2(PSx_{2n} - Tx_{2n+1}),$
$$N_2(S^2x_{2n} - Qx_{/}2n + 1))\}.$$

Letting $n \to \infty$ on both sides we have

$N_2(Sz - z) \leq K N_2(Sz - z)$ which is a contradiction. Hence we have $Sz = z$. Similarly,

$$N_2(Pz - Qx_{2n+1}) < K \max\{N_2(Sz - Tx_{2n+1}), N_2(Pz - Sz),$$
$$N_2(Qx_{2n+1} - Tx_{2n+1}), N_2(Pz - Tx_{2n+1}), N_2(Sz - Qx_{2n+1})\}.$$

Letting $n \to \infty$ on both sides, we get $z = Pz$.

Thus we have $z = Pz = Sz$. It shows that there exists a point z' in X such that $Sz = z = Pz = Tz'$, since the range of P is contained in the range of T.

Now, $QTz' = Qz = TQz'$ ... (A) as Q and T commute. Further $z = Qz'$ for

$$N_2(z - Qz') = N_2(Pz - Qz') < K \max\{N_2(Sz - Tz'), N_2(Pz - Sz),$$
$$N_2(Qz' - Tz'), N_2(Pz - Tz'), N_2(Sz - Qz')\} = K N_2(z - Qz'),$$

which implies $z = Qz'$, By (A) and $z = Qz'$, $Qz = Tz$. Thus $z = Qz = Tz$ for otherwise

$$N_2(z - Qz) = N_2(Pz - Qz) < K \max \{N_2(Sz - Tz), N_2(Pz - Sz),$$
$$N_2(Qz - Tz)\, N_2(Pz - Tz), N_2(Sz - Qz)\}$$
$$= K\, N_2(z - Qz) \text{ which implies } z = Qz.$$

Thus we have $z = Pz = Sz = Qz = Tz$. So far we have proved that z is a common fixed point of P, Q, S and T. The proof is similar if T is continuous instead of S.

Now suppose that the mapping is continuous. Since P and S commute the sequences $\{P^2x_{2n}\}$ and $\{SPx_{2n}\}$ converge to Pz. So,

$$N_2(P^2x_{2n} - Qx_{2n+1}) < K \max \{N_2(SPx_{2n} - Tx_{2n+1}), N_2(P^2x_{2n} - SPx_{2n}),$$
$$N_2\, Qx_{2n+1} - Tx_{2m+1}, N_2(P^2x_{2n} - T\,x_{2n+1}), N_2(SPx_{2n} - Qx_{2n+1})\}$$

on letting $n \to \infty$ on both sides will lead to $z = Pz$. This shows there exists a point z' in X such that $z = Pz = Tz'$ as we have $z \in PX$ and since $PX \subset TX$, Then we have,

$$N_2(P^2x_{2n} - Qz') < K \max \{N_2\, (Sx_{2n} - Tz'), N_2(P^2x_{2n} - SPx_{2n}), N_2(Qz' - Tz'),$$
$$N_2(P^2x_{2n} - Tz'), N_2(SPx_{2n} - Q(z')\}$$

on letting $n \to \infty$ on both sides, we obtain $z = Qz'$

i.e. $Tz = TQz' = QTz' = Qz$. Further

$$N_2(Px_{2n} - Qz) < K \max \{N_2\, (Sx_{2n} - Tz), N_2(Sx_{2n} - Px_{2n}), N_2(Qz - Tz),$$
$$N_2(Px_{2n} - Tz), N_2(Sx_{2n} - Qz)\}$$

Again on letting $n \to \infty$ we get, $z = Qz = Tz$. This shows that there exists a point z'' in X such that $z = Qz = Sz''$ with the same reasoning that $QX \subset SX$. Now,

$$N_2(PZ'' - z) = N_2(Pz'' - Qz) < K \max \{N_2(Sz' - Tz),$$
$$N_2(Pz'' - Sz''), N_2(Qz - Tz), N_2(Pz'' - Pz), N_2(Sz'' - Qz)\}.$$

From which again we get $z = Pz'' = Sz''$. Since P and S commute $PSz'' = Sz = SPz'' = Pz = z$. Thus we have proved that z is again a common fixed point of P, Q, S and T. If the mapping Q is continuous instead of P, then the proof that z is again a comon fixed point of P, Q, S and T is similar.

Now we shall show the uniqueness of the point z. Suppose that w is also a common fixed point of P and S. Then,

$$N_2(w - z) = N_2(Pw - Qz) < K \max \{Sw - Tz), N_2(Pw - Sw), N_2(Qz - Tz),$$
$$N_2(Pw - Tz)\} = K\, N_2(w - z)$$

which leads us to conclude $w = z$. Thus z is the unique common fixed point of P and S. Similarly we can prove that z is the unique common fixed point of Q and T.

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ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BELONGING TO THE LIPSCHITZ CLASS BY $F(a, q)$ MEANS

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ABSTRACT

In the present paper, we obtain the degree of approximation of $f \in \text{Lip}\,\alpha$ $(0 < \alpha \leq 1)$ by $F(a, q)$ means of its Fourier series.

1. INTRODUCTION

Let $C_{2\pi}$ be the space of all 2π - periodic and continuous functions defined on $[-\pi, +\pi]$, which is Banach space under "Sup" norm. For each $f \in C_{2\pi}$, let the Fourier series be given by

$$(1.1) \qquad f(x) \sim \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x).$$

We write

$$(1.2) \qquad \Phi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}.$$

The family $F(a, q)$ of summability methods was introduced by Meir [5]. The summability matrix $\{C_{pk}\}$ belongs to $F(a, q)$ if it satisfies the following conditions :

p is a discrete or continuous parameter : $q = q(p)$ is a positive increasing function which tends to infinity as $p \to \infty$; 'a' is a positive constant : for every fixed $\delta : 1/2 < \delta < 2/3$

$$(1.3) \qquad C_{pk} = \sqrt{\frac{a}{\pi q}}\ \exp\left(-aq^{-1}(k-q)^2\right)$$
$$\left\{1 + 0\left(\frac{|k-q|+1}{q}\right) + 0\left(-\frac{|k-q|^3}{q^2}\right)\right\}$$

as $p \to \infty$ uniformly in k for $|k - q| \leq q^{\delta}$; and

$$(1.4) \qquad \sum_{|k-q| > q^{\delta}} (k+1)\, C_{pk} = 0\,\{\exp(-q^{\mu})\},$$

where μ is some positive number independent of p.

Let

$$(1.5)\qquad t_p(f:x)=\sum_{k=0}^{\infty} C_{pk}\, s_k\,(f:x)$$

denote the $F(a, q)$ mean of the Fourier series (1.1) of f, where $s_k(f:x)$ is is the kth partial sum of (1.1).

The family $F(a, q)$ contains the summability methods of generalised Borel, Euler, Taylor, S_α and (e, c).

It is known (see [4]) that

$$(1.6)\qquad \sum_{k=0}^{\infty} C_{pk}=1+0\,(q^{-1/2}).$$

The summability methods of Euler, Taylor, S_α and Borel satisfy (1.6) in the stronger form

$$(1.7)\qquad \sum_{k=0}^{\infty} C_{pk}=1.$$

2. Degree of approximation by borel means and (E, q) - means were obtained by Chandra [1] and [2] respectively. Extending the results of Chandra to (e, c) mean we [6] have proved the following theorems :

THEOREM A. Let $f\in C_{2\pi}\cap \text{Lip}\,\alpha$, $0<\alpha\le 1$. Then

$$(2.1)\qquad \|t_n^c-f\|=0\,(n^{-\alpha/2}),$$

where $t_n^c\,(f; x)$ is n^{th} $(e;, c)$ means of the Fourier series of f at x.

Since $F(a, q)$ method includes (e, c) method, it is natural to ask as to what will be the result if we apply $F(a, q)$ mean to obtain the degree of approximation for $f\in \text{Lip}\,\alpha$ $(0<\alpha\le 1)$?

We shall prove the following theorem :

THEOREM : Let $[q]$ denote the integral part of $q=q(p)$ and $m=[q]+1$. If $f\in \text{Lip}\,\alpha$, $0<\alpha\le 1$, then

$$(2.2)\qquad \|t_p(f\,;x)-f(x)\|=\sup_{-\pi\le x\le\pi}|t_p(f:x)-f(x)|=0(m^{-\alpha/2}).$$

3. For the proof of our theorem we shall need the following lemmas :

LEMMA 1. If $q=q(p)$, is an integer valued function of p, then, for $\frac{1}{2}<6<\frac{2}{3}$ we have

$$\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\sum_{|k-q|\le q^{\delta}}\sqrt{\frac{a}{\pi q}}\,\exp\,(-aq^{-1}\,(k-q)^2)\,\sin\,(k+1/2)t\,dt$$

$$=\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\exp\left(-\frac{qt^2}{4a}\right)\sin\left(q+\frac{1}{2}\right)t\,dt+O(q\,\exp\,(-aq^{2\delta-1})).$$

PROOF OF LEMMA 1. From the proof of Lemma 3.2 of Ikeno [3] we have

$$\sum_{|k-q|\le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\sin(k+1/2)t$$
$$= \exp(-\frac{qt^2}{4a})\sin(q+\frac{1}{2})t\,dt + O(q\exp(-aq^{2\delta-1})\ |t|).$$

Hence the Lemma 1 follows in view of boundedness of $\Phi_x(t)$ and since $\sin(t/2) > t/\pi$ $(0 < t < \pi)$.

LEMMA 2. If m is as defined in the theorem, we have

$$\int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{|k-q|\le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\sin(k+1/2)t\,dt$$
$$= \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{|k-m|\le m^{6}} \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2)\sin(k+1/2)t\,dt$$
$$+ O(m^{-\alpha})$$

PROOF OF LEMMA 2. We estimate the difference :

$$\int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{|k-q|\le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\sin(k+1/2)t\,dt$$
$$-\int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{|k-m|\le m^{\delta}} \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2)\sin(k+1/2)t\,dt$$
$$= \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{m\le k\le m+m^{\delta}} \Bigg[\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$
$$- \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m^2)\Bigg]\sin(k+1/2)t\,dt$$
$$+\int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \sum_{m-m^{\delta}\le k< m} \Bigg[\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$
$$- \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m^2)\Bigg]\sin(k+1/2)t\,dt$$
$$-\int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{q+q^{\delta}<k\le m+m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\sin(k+1/2)t\,dt$$
$$+\int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{q-q^{\delta}\le k\le m-m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$

$$\sin (k + 1/2)t\, dt$$

(3.1) $= D_1 + D_2 + D_3 + D_4.$

From Ikeno [3], p. 261, 262, we have

$$\sum_{q+q^{\delta} < k \le m+m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2).\sin(k+1/2)t\, dt$$
$$= O(\sqrt{q}\, t \exp(-aq^{2\delta-1}))$$

and also

$$\sum_{q-q^{\delta} \le k \le m-m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\sin(k+1/2)t\, dt$$
$$= O(\sqrt{q}\, t \exp(-aq^{2\delta-1}))$$

Thus,

$$|D_3| \le \int_0^{\pi} \frac{\|\Phi_x(t)\|}{t} \sum_{q+q^{\delta} < k \le m+m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\, |\sin(k+1/2)t|\, dt$$

(3.2) $$= O(\sqrt{q} \exp(-aq^{2\delta-1})).$$

Similarly,

(3.3) $$|D_4| = 0(\sqrt{q} \exp(-aq^{2\delta-1})).$$

In case where $q < m \le k \le m + m^{\delta}$, we have

$$0 \le \frac{(k-m)}{\sqrt{m}} < \frac{(k-q)}{\sqrt{q}} < \frac{(k-[q])}{\sqrt{[q]}}.$$

Therefore, from Ikeno [3], p. 259

(3.4) $$\left| \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) - \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \right|$$
$$= O\{\frac{1}{\sqrt{m}} \exp(-am^{-1}(k-m)^2)(\frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m})\}.$$

Using (3.4), we obtain

$$|D_1| = O\ [\int_0^{\pi} \frac{|\Phi_x(t)|}{t} \sum_{m \le k \le m+m^{\delta}} \{\frac{1}{\sqrt{m}} \exp(-am^{-1}(k-m)^2)$$
$$(\frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m})\}.\ |\sin(k+1/2)t|\ dt]$$

$$= 0\ [\int_0^{\pi/m} \frac{|\Phi_x(t)|}{t} \sum_{m \le k \le m+m^{\delta}} \{\frac{1}{\sqrt{m}} \exp(-am^{-1}(k-m)^2)$$

$$\left(\frac{(k-m)^2}{m^2}+\frac{|k-m|}{m}+\frac{1}{m}\right)\}.\ (|k-m\ |+m+1/2)t\,dt$$

$$+O\ [\int_{\pi/m}^{\pi}\frac{|\phi_x(t)|}{t}\sum_{m\le k\le m+m^{\delta}}\{\frac{1}{\sqrt{m}}\exp(-am^{-1}(k-m)^2)$$

$$\left(\frac{(k-m)^2}{m^2}+\frac{|k-m|}{m}+\frac{1}{m}\right)\}dt]$$

$$=O\,\{\sqrt{m}\int_0^{\pi/m}|\phi_x(t)|\,dt\}+0\,\{\frac{1}{\sqrt{m}}\int_{\pi/m}^{\pi}\frac{|\phi_x(t)|}{t}\,dt\}$$

(3.5) $\quad = O(m^{-\alpha})$.

Also in case where $k \le [q] < q < m$, we have

$$\frac{(k-m)}{\sqrt{m}}<\frac{(k-q)}{\sqrt{q}}<\frac{(k-[q])}{\sqrt{[q]}}<0.$$

Hence, from Ikeno [3, p. 260] following estimate results :

$$\left|\sqrt{\frac{a}{\pi q}}\exp(-aq^{-1}(k-q)^2)-\sqrt{\frac{a}{\pi m}}\exp(-am^{-1}(k-m)^2)\right|$$

$$=O\,\{\frac{1}{\sqrt{[q]}}\exp(-a[q]^{-1}(k-[q])^2)\left(\frac{(k-[q])^2}{[q]^2}+\frac{|k-[q]|}{[q]}+\frac{1}{[q]}\right)).$$

Proceeding as in the estimation of D_1 and using the above inequality we get

(3.6) $\quad |D_2| = O(m^{-\alpha})$.

Thus, Lemma 2 follows from (3.1) to (3.6).

4. PROOF OF THE THEOREM.

Since $S_k(f\,;x)-f(x)=\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\frac{\phi_x(t)}{\sin t/2}\sin(k+1/2)t\,dt$

we have

$$t_p(f\,;x)-f(x)=\frac{1}{\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin t/2}\sum_{k=0}^{\infty}C_{pk}\sin(k+1/2)t\,dt+O(q^{-1/2})$$

$$=\frac{1}{\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin t/2}\,[\,(\sum_{|k-q|\le q^{\delta}}+\sum_{|k-q|\le q^{\delta}})C_{pk}\sin(k+1/2)t]dt+O(q^{-1/2})$$

(4.1) $\quad = S_1 + S_2 + 0(q^{-1/2})$.

By (1.4) and the fact that $\sin t/2 \ge t/\pi$ $(0 < t \le \pi)$, we have

$$|S_2|\le\frac{|\phi_x(t)|}{t}\sum_{|k-q|>q^{\delta}}C_{pk}\,(k+1/2)t\,dt$$

$$(4.2) \qquad = O(\exp(-q^{\mu})).$$

Now making use of (1.3), we can write

$$S_1 = \frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$

$$\{1 + O(\frac{|k-q|+1}{q}) + O(\frac{|k-q|^3}{q^2})\}. \sin(k+1/2)t\, dt$$

$$(4.3) \qquad = S_3 + S_4 + S_5.$$

We estimate S_4 as follows :

$$|S_4| \le \int_0^{\pi} \frac{|\phi_x(t)|}{t} \sum_{|k-q| \le q^{\delta}} \{\sqrt{\frac{a}{\pi q}} \exp - (aq^{-1}(k-q)^2)$$

$$O(\frac{|k-q|+1}{q})\} \,|\sin(k+1/2)t|\, dt$$

$$= \int_0^{\pi/q} \frac{|\phi_x(t)|}{t} \sum_{|k-q| \le q^{\delta}} \{\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$

$$O(\frac{|k-q|+1}{q})\} \,(|k-q| + q + 1/2)t\, dt$$

$$+ \int_{\pi/q}^{\pi} \frac{|\phi_x(t)|}{t} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$

$$O(\frac{|k-q|+1}{q})\, dt$$

$$= O(\sqrt{q} \int_0^{\pi/q} t^{\alpha}\, dt)\, O(\frac{1}{\sqrt{q}} \int_{\pi/q}^{\pi} t^{\alpha-1}\, dt)$$

$$(4.4) \qquad = O(q^{-\alpha}).$$

Similarly,

$$(4.5) \qquad |S_5| = O(q^{-\alpha}).$$

Applying Lemma 1 and Lemma 2 and noting that $m = m(p)$ is an integer valued function of p we get

$$S_3 = \int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \exp(-\frac{mt^2}{4a}) \sin(m+1/2)t\, dt.$$

Now

$$|S_3| = O(1) \int_0^{\pi} t^{\alpha-1} \exp(-\frac{mt^2}{4a})\, |\sin(m+1/2)t|\, dt$$

$$= O(1) \int_0^{\pi/\sqrt{m}} t^{\alpha-1}\, dt + O(m^{-1}) \int_{\pi/\sqrt{m}}^{\pi} t^{\alpha-2} \frac{\partial}{\partial t} (\exp(-\frac{mt^2}{4a}))dt$$

(4.6) $= O(m^{-\alpha/2})$.

Collection of (4.1), (4.2) , ..., (4.6) completes the proof of the Theorem.

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