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INTEGRATION OF *I*-FUNCTION OF TWO VARIABLES WITH RESPECT TO PARAMETERS

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ABSTRACT

The object of the present paper is to establish four integrals associated with *I*-function of two variables due to Goyal and Agrawal [3]. The integration is performed with respect to a parameter. Such integrals are usuful in the study of certain boundary value problems.

1. INTRODUCTION

The *I*-function of two variables represented by means of two contour integrals is defined by Goyal and Agarwal [3] and represented in as following manner :

$$I\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \frac{1}{(2\pi w)^2} \int_{l_1} \int_{l_2} \Phi_1(\xi)\, \Phi_2(\eta)\, \Psi(\xi, \eta)\, z_1^{\xi} z_2^{\eta}\, d\xi\, d\eta$$

where

$$w = \sqrt{-1},$$

$$\Phi_1(\xi) = \frac{\prod_{j=1}^{m_2} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{n_2} (-a_j + \alpha_j \xi)}{\sum_{i=1}^{r} \left[\prod_{j=m_2+1}^{q_i^{(1)}} \Gamma(1 - b_{ji} + \beta_{ji}\xi) \prod_{j=n_2+1}^{p_i^{(1)}} \Gamma(a_{ji} - \alpha_{ji}\xi) \right]} \quad \text{... (1.2)}$$

$$\Phi_2(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(d_j - \delta_j \eta) \prod_{j=1}^{n_3} \Gamma(1 - c_j + \gamma_j \eta)}{\sum_{i=1}^{r} \left[\prod_{j=m_3+1}^{q_i^{(2)}} \Gamma(1 - d_{ji} + \delta_{ji}\eta) \prod_{j=n_3+1}^{p_i^{(2)}} \Gamma(c_{ji} - \gamma_{ji}\eta) \right]} \quad \text{... (1.3)}$$

$$\Psi(\xi,\eta)=\frac{\prod_{j=1}^{m_1}\Gamma(f_j-F_j(\xi+\eta))\prod_{j=1}^{n_1}\Gamma(1-e_j+E_j(\xi+\eta))}{\prod_{j=m_1+1}^{q}\Gamma 1-f_j+F_j(\xi+\eta\prod_{j=n_1+1}^{p}\Gamma e_j-E_j(\xi+\eta)} \quad \text{... (1.4)}$$

z_1 and z_2 both are not equal to zero.

$$A=\sum_{j=1}^{n_1}E_j-\sum_{j=n_1+1}^{p}E_j+\sum_{j=1}^{m_1}F_j-\sum_{j=m_1+1}^{q}F_j+\sum_{j=1}^{m_2}\beta_j-\sum_{j=m_2+1}^{q_i^{(1)}}B_{ji}$$

$$+\sum_{j=1}^{n_2}\alpha_j-\sum_{j=n_2+1}^{p_i^{(1)}}\alpha_{ji}>0 \quad \text{... (1.5)}$$

$$B=\sum_{j=1}^{n_1}E_j-\sum_{j=n_1+1}^{p}E_j+\sum_{j=1}^{m_1}F_j-\sum_{j=m_1+1}^{q}F_j+\sum_{j=1}^{m_3}\delta_j-\sum_{j=m_3+1}^{q_i^{(2)}}\delta_{ji}$$

$$+\sum_{j=1}^{n_3}\gamma_j-\sum_{j=n_3+1}^{p_i^{(2)}}\gamma_{ji}>0 \quad \text{... (1.6)}$$

$$\forall\ i=1,2,\ldots r$$

$$|arg\, z_1|<\frac{A\pi}{2},\quad |arg\, z_2|<\frac{B\pi}{2}.$$

For asymptotic expansion and all other conditions, see Goyal Anil and Agrawal R.D. [3]. These conditions will be assumed to hold good in the I-function of two variables occuring in this paper.

2. FORMULAE REQUIRED

In this paper, we require the following results

It is a hypergeometric function with unit argument given by Whittakar and Watson [2]

$$\frac{1}{2\pi w}\int_{-i\infty}^{+i\infty}\frac{\Gamma(a+x)\,\Gamma(b-x)\,\Gamma c-x)}{\Gamma(d-x)}\,e^{\pm i\pi x}\,dx$$

$$=\frac{\Gamma(a+b)\,\Gamma(a+c)\,\Gamma(d-a-b-c)}{\Gamma(d-b)\,\Gamma(d-c)}\,e^{\pm i\pi a}$$

$$\mathrm{Re}(d-a-b-c)>0 \quad \text{... (2.1)}$$

$$(\alpha)_n=\begin{cases}\alpha(\alpha+1)\,(\alpha+2)\ldots(\alpha+n-1), & n\geq 1\\ 1 \quad \text{for} & n=0,\ \alpha\neq 0\end{cases} \quad \text{... (2.2)}$$

3. INTEGRALS

The integrals involving the I-function of two variables to be evaluated are

(I) $$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma b-x\,\Gamma c-x}{\Gamma d-x}e^{\pm i\pi x}\; I^{m_1, n_1+1:P}_{p+1, q+1:Q}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\left|\begin{matrix}(1-a-x:h,k),(e_p,E_p):T \\ (f_q,F_q) \qquad :T^1\end{matrix}\right.\right]dx$$

$$= \frac{e^{\pm i\pi a}}{\Gamma d-b\,\Gamma d-c} I^{m_1+1, n_1+2:P}_{p+2, q+1:Q}\left[\begin{matrix} z_1 e^{\pm i\pi h} \\ z_2^{\pm i\pi k} \end{matrix}\left|\begin{matrix}(1-a-b:h,k),(1-a-c:h,k)\,(e_p,E_p):T \\ (d-a-b-c:h,k),(f_q,F_q) \qquad T^1\end{matrix}\right.\right]dx$$

... (3.1)

provided $\quad h, k > 0$

$$\mathrm{Re}\left[d-a-b-c+h \min_{1\le j\le m_2} R\left(\frac{b_i}{\beta_j}\right)+k\min_{1\le j\le m_3} R\left(\frac{d_j}{\delta_j}\right)\right]>0$$

... (3.1.1)

$$P = m_2, n_2; m_3, n_3, \quad Q = p_i^{(1)}, q_i^{(1)}; p_i^{(2)}, q_i^{(2)} : r \qquad \text{... (3.12)}$$

$$T = [(a_j, \alpha_j)_{1, n_2}], [(a_{ji}, \alpha_{ji})_{n_{2+1}}, p_i^{(1)}]; [c_j, \gamma_j)_{1, n_3}, [(c_{ij}, \gamma_{ji})_{m_3+1}, p_i^{(2)}]$$

... (3.13)

$$T^1 = [(b_j, \beta_j)_{1, m_2}, [(b_{ji}, \beta_{ij})_{m_2+1, q_i^{(1)}}]; [(d_j, \delta_j)_{1, m_3}], [(d_{ji}, \delta_{ji})_{m_3+1, q_i^{(2)}}]$$

... (3.14)

(II) $$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\Gamma(a+x)\,\Gamma(b-x)\,\Gamma c-x\, e^{\pm i\pi x}$$

$$I^{m_1, n_1:P}_{p+1, q:Q}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\left|\begin{matrix}(e_p,E_p),(d-x:h,k):T \\ (f_q,F_q) \qquad :T^1\end{matrix}\right.\right]dx$$

$$= e^{\pm i\pi a}\Gamma(a+b)\,\Gamma(a+c)\, I^{m_1+1, n_1:P}_{p+2, q+1:Q}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\left|\begin{matrix}(e_p,E_p),(d-b:h,k),(d-c:h,k):T \\ (d-a-b-c:h,k),(f_q,F_q)) \qquad :T^1\end{matrix}\right.\right]$$

... (3.2)

provided $\quad h, k > 0$

$$\mathrm{Re}\left[d - a - b - c - h \min_{1 \le j \le m_2} \mathrm{Re}\left(\frac{b_i}{\beta_j}\right) + k \min_{1 \le j \le m_3} \mathrm{Re}\left(\frac{d_j}{\delta_j}\right)\right] > 0 \qquad \ldots (3.2.1)$$

(III) $$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \Gamma(a+x)\,\Gamma(b-x)\,\Gamma(c-x)\, e^{\pm i\pi x}$$

$$I^{m_1, n_1+1 : P}_{p+2, q : Q}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\left|\begin{matrix} (1-c+x : h, k), (e_p, E_p), (d-x; h, k) : T \\ (f_q, F_q) \qquad : T^1 \end{matrix}\right.\right] dx$$

$$= e^{\pm i\pi a}\Gamma(a+b)\, I^{m_1+n_1+1 : P}_{p+3, q+1 : Q}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\left|\begin{matrix} (1-a-c : h, k), (e_p, E_p), (d-b : h, k), (d-c : 2h, 2k) : T \\ (d-a-b-c : 2h, 2k), (f_q, F_q)) \qquad : T^1 \end{matrix}\right.\right] \qquad \ldots (3.3)$$

provided $h, k > 0$

$$\mathrm{Re}\left[d - a - b - c - 2h \min_{1 \le j \le m_2} R\left(\frac{b_j}{\beta_j}\right) - 2k \min_{1 \le j \le m_3} R\left(\frac{d_j}{\delta_j}\right)\right] > 0 \qquad \ldots (3.3.1)$$

(IV) $$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(a+x)\,\Gamma(b-x)}{\Gamma(c-x)} e^{\pm i\pi x}\, I^{m_1, n_1+1 : P}_{p+1, \; q \; : Q}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\left|\begin{matrix} (1-d-x : h, k), (e_p, E_p) : T \\ (f_q, F_q) \qquad : T^1 \end{matrix}\right.\right] dx$$

$$= e^{\pm i\pi a}\frac{\Gamma(a+b)}{\Gamma(c-b)}\, I^{m_1+1, n_1+1 : P}_{p+2, q+1 : Q}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\left|\begin{matrix} (1-a-d) : h, k), (e_p, E_p), (c-d : h, h, k) : T \\ (c-a-b-d : h, k), (f_q, F_q) \qquad T^1 \end{matrix}\right.\right] dx \qquad \ldots (3.4)$$

provided $h, k > 0$

$$\mathrm{Re}\left[c - a - b - d - h \min_{1 \le j \le n_2} R\left(\frac{a_j - 1}{\alpha_j}\right) - k \max_{1 \le j \le n_3} R\left(\frac{c_j - 1}{\gamma_j}\right)\right] > 0 \qquad \ldots (3.4.1)$$

Here P, Q, T, T^1 are defined as per equations (3.1.2), (3.1.3), (3.1.4) respectively.

Proof. To prove the integral (3.1), we replace I-function of two variables by (1.1), change the order of integration which is justified under the conditions stated in (1.5), (1.6). We get

$$\frac{1}{(2\pi w)^2}\int_{i_1}\int_{i_2} \Phi_1(\xi)\,\Phi_2(\eta)\,\psi(\xi, n)\, z_1^{\xi} z_2^{\eta}$$

$$=\left\{\frac{1}{2\pi i}\int_{-ix}^{+i\infty}\frac{\Gamma b-x\,\Gamma a-x+h\xi+k\eta\,\Gamma c-x}{\Gamma d-x}\,e^{\pm i\pi x}\,dx\right\} d\xi\, d\eta \quad \ldots (3.5)$$

Now we evaluate the inner integral using a known integral formula given by Whittakar and Watson [2], noting that it is a hypergeometric function with unit argument.

On expressing the resulting expression with the help of (1.1) we obtain (3.1).

Proceeding similarly we easily evaluate the integrals (3.2), (3.3), (3.4) respectively.

PARTICULAR CASES

(i) If we take $r = 1$, all greek letters equal to unity in (1.1), the I-function of two variables would reduce to relatively more familiar G-function of two variables. Thus our result (3.1), (3.2), (3.3), (3.4) will yield similar integrals involving the G-function of two variables.

(ii) If we set $r = 1, m_1 = 0$ in (3.1), (3.2), (3.3), (3.4), we get integrals for the H-function of two variables.

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