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SCATTERING IN GENERALIZED FUNCTION SPACE

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1. INTRODUCTION

The most direct information about the nature of forces between the particles is obtained from the study of collisions. The experimental data can be interpreted in terms of the model of the miscroscopic details of a single collision and the conclusions concerning the interaction between the scattered particle and the scatter can be drawn. In recent years this study has assumed quite importance because of its wide applications.

Attempts have been made [1, 4, 5] to analyse this type of scattering by several authors but all of them have givn aproximate methods and adoption of these analytical theories makes it difficult to choose the appropriate value of parametres thereby restricting the proper choice of energy values of the system and the symmetry of the potential function.

The present paper attempts to solve the corresponding differential equation in a general approximation free way in the so called generalized function space (or distribution space). The technique adopted substantially expands the range of problems that can be tackled.

In what follows it is assumed that the interaction between the scattered particle and the scatterer can be represented by the potential energy function $V(\boldsymbol{r})$, where $\boldsymbol{r}$ is the vector joining the scattered particle and the centre of force. At present we restrict ourselves to elastic scattering only in which the kinetic energy of the system is not changed due to the collision. If the mass of the scatterer is large as compared to that of scattered particle e.g. in the collision between an electron and an atom the scatterer can be assumed to remain at rest during the entire collision process.

2. Notation and Terminology

A generalized function is a generalization of the classical notion of a function. It reflects the fact that in reality one can not measure the value of a physical quantity at a point, but can only measure the mean values within sufficiently small neighbourhoods of the point and then proclaim the limit of the sequence of those mean values as the value of the physical quantity at the given point. Thus it enables one to express in regorous mathematical form such ideal concepts as density of a point charge or dipole and the Paul Dirac's delta function.

We shall be interested in finding the solution of the differential equation of the scattering process discussed in the last para of Section 1, viz.

$$(2.1) \qquad (\nabla^2 + k^2) \; = \frac{2m}{h^2} V(\boldsymbol{r}),$$

where $k^2 = \dfrac{2m}{h^2} E$

in the space τ[2] of basic functions i.e. the space of infinitely differentiable functions which, together with their derivatives, aproach zero more rapidly than any power of $1/|x|$ as $|x| \to \infty$. The generalized function space of τ i.e. the set of continuous linear functionals on τ shall be denoted by τ' as dual space of τ. The space τ' is known as the space of generalized functions of slow growth and is very important in applications of Mathematical Physics. The terminology used here is that of [2].

3. Formulation of the problem

In general, the potential energy $V(\boldsymbol{r})$ decreases in magnitude as the distance $r = |\hat{\boldsymbol{r}}|$ from the scattering centre becomes large, it is convenient to choose the arbitrary constant in the definition of $V(\boldsymbol{r})$ such that $V = 0$ at $r = \infty$.

If $V(r)$ decreases to zero sufficiently rapidly as $r \to \infty$, the particle can be considered essentially free when r is large. The asymptotic wave function is therefore, in general, a linear superposition of the free particle wave function *exp* $(i\boldsymbol{k}.\boldsymbol{r})$.

A solution of (2.1) is sought which satisfies the boundary condition

$$(3.1) \qquad \psi \sim \exp(i\,\hat{\boldsymbol{k}}.\,\hat{\boldsymbol{r}}) + f(k, k')\, eikr/r$$

,where r is large enough for the particle to be beyond the range of force. The scattering amplitude is $f(k, k')$, where k and k' are propogation vectors.

It is convenient to separate the incidental wae function *exp* $(i\,\boldsymbol{k}.\boldsymbol{r})$ from the wave function and to write

$$(3.2) \qquad \psi = \exp(i\,\hat{\boldsymbol{k}}\,.\,\hat{\boldsymbol{r}}) + \psi_s$$

where ψ_s the scatterer wave, is asymptotic to e^{ikr}/r. Since $\exp(i\,\hat{\boldsymbol{k}}\,.\,\hat{\boldsymbol{r}})$ is a solution of homogeneous wave equation

$$(3.3) \qquad (\nabla^2 + k^2) \exp(i\,\hat{k}\,.\,\hat{r}) = 0,$$

the corresponding Schrodinger's equation takes the inhomogeneous form

(3.4) $(\nabla^2 + k^2)\, \psi_s = \frac{2m}{h^2} V(r)\psi.$

A formal solution of (3.4) can now be obtained in generalized function space τ'.

4. The solution in the space τ' .

Let us take

(4.1) $\frac{2m}{h^2} - V(r)\psi = -\rho(r)$

Here the quantity $\rho(r)$ can be regarded as a source density function for scattered divergent spherical waves.

Before finding the actual solution it may be pointed out that if $\psi(s_1)$ and $\psi(s_2)$are solutions of (3.7), with $\rho(r)$ as in (4.1), belonging to density functions $\rho_1(r)$ and $\rho_2(r)$ satisfying

(4.2) $< \psi s_1, \phi > < f_1, e^{ikr}/r, \ \phi>$, and

(4.3) $< \psi_2\, \phi > \sim < f_2\, e^{ikr}/r, \phi>$

respectively; then $\psi_s = \psi_{s_1} + \psi_{s_2}$ is a soluiton belonging to $f(r) = \rho_1(r) + \rho_2(r)$ such that

(4.4) $< \psi_S, \phi > \sim < f, e^{ikr}/r, \phi>,$

where $f = f_1 + f_2$ is a well defined linear continuous functional defined over the space of density functions $\rho_1(r)$ and $\rho_2(r)$.

Here

(4.5) $< f, \phi >$

is taken to mean as the application of the functional $f \in \tau'$ to the test function $\phi \in \tau$.

Also the linearity of f is signified by the relation

(4.6) $< f, \alpha_1\phi_1 + \alpha_2\phi_2> = < \alpha_1 ; f, \phi_1 > + < \alpha_2\ f, \phi_2>$

where as continuity of the generalized function f implies that if the sequence $\phi($, ϕ_2, ..., ϕ_v, ... converges to zero in τ then the sequence $< f, \phi_1>$, $<f, \phi_2>$, ..., $<f_1, \phi_v>$, ... also converges to zero.

For *regular* functionals

4.7 $< f, \phi> = \int f(x)\, \phi\, (x)\, dx.$

Functionals that can not be represented as (4.7) are called *singular*.

For example the delta functional represented by

(4.8) $< \delta(x), \phi(x) > = \phi(0)$

is singular, and so is the socalled "translated" delta functional defined by

(4.9) $< \delta\,(x - x_0), \phi(x)> = \phi(x_0)$.

Now to obtain the desired solution let $E(r, r')$ be the solution of the equation

(4.10) $(\nabla^2 + k^2)\, E(r, r') = -\, \delta(r, r')$,

where $E(r, r')$ is asymptotic to e^{ikr}/r.

Here $\delta(r, r')$ is the source density of unit strength.

Using the distributional Fourier transform technique, we have

(4.11) $E(r, r') = \dfrac{1}{2\pi} < E(r, r'), e^{-ik(r - r')\xi}>$,

where $E(r, r')$ denotes the distributional Fourier transform of

$E(r, r')$ *i.e.* $E(r, r') = 1/k^2 - \xi^2$, using (4.10).

The solution for the scattering problem for the density function $\rho(r)$ can then be calculated from the equation

(4.12) $\psi_s = \rho(r)^*\, E(r, r')$.

Here $*$ denotes the convolution defined by

(4.13) $< f * g, \phi > = < f(x) \times g(y), \phi\,(x + y)>$.

For example

$< \delta * f, \phi > = < \delta(x) \times f(y), \phi(x + y) >$

$= < f(y), < \delta\,(x), \phi(x + y)>> = < f(y), \phi(y)> = < f, \phi>$.

Thus for any generalized function f and the generalized delta function defined by (4.9), we have

(4.14) $\delta * f = f * \delta = f$

and that, in general,

(4.15) $f * g = g * f$

holds also for generalized functions f and g in which the convolution operation has a meaning.

Now, using (4.9), the arbitrary density $\rho(r)$ is given by

(4.16) $\rho(r) = < \delta(r, r'),\ \rho(r')>$,

where $\rho(r')$ is the density function of point source at r'.

The scattering problem is thus formulated in the form of an integral equation, by (4.1) and (3.2)

(4.17) $\psi(r) = \exp\,(i\, \hat{k}\, .\, \hat{r}) - \dfrac{2m}{h^2} < E(r, r'), V(r')\ \psi\,(r')>$

5. Discussion

Equation (4.17) shows that the scattered wave at the point r is composed of spherical waves $E(r, r')$ originating at each point of space

r. The amplitude of each contribution is proportional to the product $V(r')\,\psi(r')$ i.e. it is porportional jointly to the strength of the interaction and the amplitude of wave function at r'. All these spherical waves are compounded at the point r and generate the total scattered wave, which is then added to the incident wave to produce the total wave function at r.

Also, if the potential energy function is assumed to be confined to a limited region of space, the asymptotic form of $E(r, r')$ as indicated at (4.10) can be substituted in (4.17) to yield the scattering amplitude

$$(4.18)\quad f(k, k') = -\frac{2m}{h^2} < \exp(-i\boldsymbol{k}\,.\,\boldsymbol{r}'),\, V(r')\,\psi(r')>$$

Equation (4.18) is a general formula by which the scattering cross section $|f(k, k')|^2$ can be computed.

Conclusion

The solution thus achieved are more general in nature than those found in the existing literature as they are neither subjected to any restriction on the energy values of the system nor on the symmetry the potential function. More over the results may find applications in the study of matter in bulk. Although, (4.17) does not provide an explicit solution as appears inside, yet the formulation of the problem according to (4.17) provides a single equation which expresses both the content of the differential equation (2.1) and the boundry condition (3.1).

Also the method applies equally suitably, if instead of the distributional Fourier transform one takes its wider generalization (introduced by the first author recently in 3), called S.M. Joshi generalized Fourier transform, given by

$$(5.1)\qquad (S_b^a f)(x) = \frac{\Gamma(a)\,\Gamma(b-\alpha)}{\Gamma(b)\Gamma(a-\alpha)} < f(\xi),\, {}_1F_1(a;b;i\,\xi\,x)>,$$

where $a = \alpha + \lambda + 1$, $b = a + \mu$, and ${}_1F_1$ is the confluent hypergeometric function. For $\mu = 0 = \alpha$, (5.1) reduces to the distributional Fourier transform

$$(5.2)\qquad (S_{b_0}^{a_0} f)(x) = <f, e^{i\xi x}> .$$

Last, but the most fruitful, advantage of the method is the fact that it applies to both regular (the classical one) and singular functional, as pointed out in (4.7) and the line following it, and it utilizes the most modern concept of generalized delta function, which existed in the literature previously only in mathematically non rigorous form-a form about which its originator Paul Dirac (See "The Principles of Quantum Mechanics, "Clarendon Press, Oxford (1958) by Dirac) himself had doubts that whether it was a function in the classical meaning.

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