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SOME FIXED POINT THEOREMS IN COMPLETE METRIC SPACES

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ABSTRACT

In this paper we prove fixed point theorems in a complete metric space which extend the theorem 2 of Khan, Swaleh and Sessa [1] and theorem 2 of H.K. Pathak and Rekha Sharma [2].

Introduction

Let R^+ be the set of non-negative real numbers and N the set of positive integers. Khan, Swaleh and Sessa [1] have established fixed point theorems for self maps of complete metric spaces by altering the distance between the points with the use of a function $\psi : R^+ \to R^+$ satisfying the following properties :

(1) ψ is continuous and increasing in R^+;

(2) $\psi(t) = 0$ if and only if $t = 0$.

We denote the set of above function ψ with φ.

In [1, Th.2] the following theorem was proved :

Theorem 1. Let (x, d) be a complete metric space, T a self map of X, and $\psi : R^+ \to R^+$ an increasing, continuous functions satisfying property (2).

Further more, let a, b, c, be three decreasing functions from $R^+/\{0\}$ into [0, 1] such that $a(t) + 2b(t) + c(t) < 1$ for every $t > 0$. Suppose that T satisfies the following condition :

(A) $$\psi(d(Tx, Ty)) \leq a(d(x,y))\, \psi(d(x,y)) + b(d(x,y))\, \{\psi(d(x, Tx)) + \psi(d(y, Ty)\} + c(d(x,y)) \min\{\psi(d(x, Ty)), \psi(d(y, Tx))\}$$

where $x, y \in X$ and $x \neq y$. Then T has a unique fixed point.

In (2, Th. 2) the following theorem was proved :

Theorem 2. Let (X, d) be a complete metric space, T a selfmap of X, and $\psi : R^+ \to R^+$ an increasing, continuous function satisfying property (2). Further, let a, b be two decreasing functions from $R^+/\{0\}$ into $[0, 1[$ such that $a(t) + b(t) < 1/2$ for every $t > 0$.

Suppose that T satisfies the following condition :

(B) $\psi(d(Tx, Ty)) \leq a(d(x, y)) \{\psi(d(x, y)) + c[\psi(d(x, y)) \psi(d(y, Tx)]^{1/2}\}$
$+ b(d(x, y) \{\psi(d(x, Tx)) + \psi(d(y, Ty))$

where $x, y \in X$ and $c \in [0, 1]$ such that $a(t). (1 + c) < 1$.

Then T has unique fixed point.

Now we establish the following theorem :

Theorem 3. Let (X, d) be a complete metric space. T is a selfmap of X, and $\psi : R^+ \to R^+$ an increasing continuous function satisfying property (2). Further let a, b be two decreasing functions from$R^+/\{0\}$ into $[0, 1[$ such that $a(t) + b(t) < 1$ for every $t > 0$. Suppose that T satisfies the following condition :

(A)′ $\psi(d(Tx, Ty) \leq a(d(x, y)) \max \{\psi(d(x, y)), [\psi(d(x, y). \psi(d(y, Tx))]^{1/2}\}$
$+ b(d(x, y) \min \{\psi(d(x, Tx), (\psi(d(y, Ty))\}$

where $x, y, \varepsilon X$. Then T has unique fixed point.

Proof: Let x_0 be a point of X. We define $x_{n+1} = Tx_n, \tau_n = d(x_n, x_{n+1})$ for all $n \in N \cup \{0\}$. We first prove that T has a fixed point. We may assume $\tau_n > 0$ for each n.

From (A)′, we obtain

$$\psi(d(Tx_n, Tx_{n+1}))$$
$$\leq a(d(x_n, x_{n+1})). \max. \{\psi(d(x_n, x_{n+1})), [\psi(d(x_n, x_{n+1})). \psi(d(x_{n+1}, Tx_n))]^{1/2}\}$$
$$+ b(d(x_n, x_{n+1})) \min \{\psi(d(x_n, x_{n+1})), \psi(d(x_{n+1}, Tx_{n+1}))\}$$

$$\Rightarrow \quad \psi(d(x_{n+1}, x_{n+2})) \leq a(\tau_n). \max \{\psi(\tau_n), [\psi(\tau n). \psi(d(x_{n+1}, x_{n+1})]^{1/2}\}$$
$$+ b(\tau_n) \min (\psi(\tau_n), \psi(\tau_{n+1})\}.$$

If $\psi(\tau_{n+1}) \leq \psi(\tau_n)$,. then we obtain

$$\psi(\tau_{n+1}) \leq a(\tau_n). \psi(\tau_n) + b(\tau_n). \psi(\tau_{n+1}).$$

(3.1) i.e. $$\psi(\tau_{n+1}) \leq \frac{a(\tau_n).}{1 - b(\tau_n)} \psi(\tau_n) < \psi(\tau_n)$$

On the other hand, if $\psi(\tau_n) \leq \psi(\tau_{n+1})$, then we have

(3.2) $$\psi(\tau_{n+1}) \leq [a(\tau_n) + b(\tau_n)] \psi(\tau_n) < \psi(\tau_n)$$

Thus $\{\tau_n\}$ is a decreasing sequence since ψ is an increasing.

In (3.1), we put $\lim_{n \to \infty} \tau_n = \tau$ and suppose that $\tau > 0$, then $\tau_n \geq \tau$ implies that

$$\psi(\tau_{n+1}) \leq \frac{a(\tau)}{1 - b(\tau)} \psi(\tau).$$

By letting $n \to \infty$, since ψ is continuous, we have

$$\psi(\tau) \leq \frac{a(\tau)}{1 - b(\tau)} \psi(\tau)$$

a contradiction. So $\tau = 0$.

Similarly if we consider (3.2) then we again have $\tau = 0$.

Now we prove that $\{x_n\}$ is a Cauchy sequence, suppose not, then there exist $\in > 0$ and two sequence $\{p^{(n)}\}, \{q^{(n)}\}$ such that for every $n \in N \cup \{0\}$, we find that

$$p^{(n)} > q^{(n)} \geq n,\ d(x_p^{(n)}, x_q^{(n)}) \geq \in \quad \text{and} \quad d(x_{p-1}^{(n)}, x_q^{(n)}) < \in.$$

For each $n \geq 0$, we put $S_n = d(x_p^{(n)}, x_q^{(n)})$. Then, we have

$$\in \leq S_n \leq d(x_{p-1}^{(n)}, x_p^{(n)}) + d(x_{p-1}^{(n)}, x_{q-1}^{(n)}) < \tau_{p-1}^{(n)} + \in .$$

Since $\{\tau_n\}$ converges to 0, so that S_n converges to $\in$..

Furthermore, the triangle inequality implies, for each $n \geq 0$,

$$-\tau_p^{(n)} - \tau_q^{(n)} + S_n \leq d(x_p^{(n)} + 1, x_p^{(n)} + 1) \leq \tau_p^{(n)} + \tau_q^{(n)} + S_n,$$

and therefore also the sequence $d(x_{p+1}^{(n)}, x_q^{(n)} + 1)$ converges to $\in$..

Now from (A) we also deduce

$$\psi(d(x_{p+1}^{(n)}, x_{q+1}^{(n)}) \leq a(d(x_p^{(n)}, x_q^{(n)}).$$

$$\max\{\psi(d(x_p^{(n)}, x_q^{(n)}), [\psi(d(x_p^{(n)}, x_q^{(n)}) \,.\, \psi(d(x_q^{(n)}, x_{p+1}^{(n)}))]^{1/2}\}$$

$$+ b(d(x_p^{(n)}, x_q^{(n)})) \min\{\psi\,(d(x_p^{(n)}, x_{p+1}^{(n)}), \psi(d(x_q^{(n)}, x_{q+1}^{(n)})$$

i.e. $\psi(d(x_{p+1}^{(n)}, x_{q+1}^{(n)}) \leq a(s_n) \max\{\psi(s_n), [\psi(s_n),$

$$\psi(s_n).\ \psi(d(x_q^{(n)}, x_{p+1}^{(n)}))]^{1/2}\} + b(s_n) \min\{\psi(\tau_p^{(n)}), \psi(\tau_q^{(n)})\}.$$

This implies that

$$\psi(d(x_{p+1}^{(n)}, x_{q+1}^{(n)}) \leq a(s_n) \max\{\psi(s_n), [\psi(s_n).$$

$$\psi((\tau_p^{(n)} + s_n)]^{1/2}\} + b\,(s_n) \min\{\psi(\tau_p^{(n)}), \psi(\tau_q^{(n)})\}.$$

Letting $n \to \infty$, we are left with

$$\psi(\in) \leq a(\in) \max\{\psi(\in), [\psi(\in).\psi(\in)]^{1/2}\} + b(\in).\ \min\{\psi(0), \psi(0)\}$$

Therefore, $\psi(\in) \leq a\ (\in)\ .\ \psi(\in) < \psi(\in)$ since $a(t) < 1$ which is absurd. Therefore $\{x_n\}$ is a Cauchy sequence.

By completness of X, $\{x_n\}$ converges to some point z. Now we show that z is a fixed point of T. Since $\tau_n > 0$ there is a subsequence $\{x_n(n)$ of $\{x_n\}$ such that $X_h(n) = z$ for each $n > 0$ and we put $\rho_n = d(z, x_n)$.

Since a, $b < 1$, we obtain from (A')

$$\psi(d(x_{h+1}(n)\ , Tz)) = \psi\ .\ (d(T\,x_h(n), Tz))$$

$$\leq a(d(Tx_h(n), z)) \max \{\psi(d(x_h(n),z)), [\psi(d(x_h(n), z)).\ \psi(d(z, x_{h+1}(n)))]^{1/2}\}$$

$$+ b((d(x_h(n), z) \min \{\psi(d(x_h(n), x_{h+1}(n))), \psi\ \{d(z, Tz))\}.$$

$$\Rightarrow \qquad \psi(d(x_{h+1}(n), Tz))$$

$$\leq a(\rho_h\ (n)).\ \max \{\psi(\rho_h\ (n)), [\psi(\rho_h\ (n)).\ \psi(d(x_{h+1}(n), z))]^{1/2}\}$$

$$+ b(\rho_h\ (n)) \min \{\psi\ \tau_h(n), \psi\ (d(z, Tz))\}$$

Since $\{\rho_n\}$ converges to 0, letting $n \to \infty$, we have

$$\lim_{n \to \infty} \psi(d(x_{h+1}(n), Tz)) < 0. \tag{3.3}$$

On the other hand the triangle inequality implies that

$$d(z, Tz) \leq \rho h\ (n) + \tau_h\ (n) + d(x_{h+1}(n)), Tz),$$

which in turn implies that

$$\psi(d(z, Tz)) \leq \lim_{n \to \infty} \psi(d(x_{h+1}(n), Tz)). \tag{3.4}$$

Therefore from (3.3) and (3.4), we have

$\psi(d(z, Tz)) = 0$ and therefore $(d(z, Tz)) = 0$.

Then z is a fixed point of T.

If T has two distinct fixed points x, y in X, then

$$\psi(d(x, y) = \psi(d(Tx, Ty))$$

$$\leq a(d(x, y)) \max \{\psi(d(x, y)), [\psi\ (d(x, y).\ \psi\ (d(y, Tx))]^{1/2}\}$$

$$+ b(d(x, y)) \min \{\psi\ (d(x, Tx)), \psi(d(y, Ty))\}$$

$$\Rightarrow \qquad \psi(d(x, y) \leq a(d(x, y)) \max \{\psi\ (d(x, y)), \psi(d(x, y))\},$$

since $Tx = x$ and $Ty = y$ $\Rightarrow \psi(d(x, y)) < \psi(d(x, y))$

a contradiction. This completes the proof.

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