Jñānābha, Vol. 25, 1995

ON A CLASS OF ENTIRE FUNCTIONS REPRESENTED BY TAYLOR DIRICHLET SERIES

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(Received : October 15, 1995)

ABSTRACT

In this paper, we have studied certain classes of Taylor Dirichlet series. The standard Taylor series and Dirichlet series turn out to be the particular cases of this series. Various functional analytic structures have been provided to these classes. Topological zero divisors, invertible elements and continuous linear functionals have been investigated for these classes. An interesting method of construction of total sets has also been formulated.

1. Introduction

In (3) Rishishwar has studied the series of type

$$f(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n \psi(z)} \tag{1.1}$$

where,

$$\lim_{n \to \infty} \sup . \frac{n}{\lambda_n} = D < \infty, \; 0 < \lambda_1 < \lambda_2 < \ldots \; (\lambda_n \to \infty \text{ as } n \to \infty).$$

$\{a_n\}$ $(n = 1, 2, 3, \ldots)$ is a sequence of complex numbers $\psi(z)$ is a function of complex variable z.

The standard Taylor series and Dirichlet series turn out to be the particular cases of the series (1.1). Hence, in future we shall call the series (1.1) as Taylor Dirichlet series (TDS-in short).

Suppose that

$$\psi(z) = \phi(-z) + i\,\theta(z)$$

where Φ are real valued functions of the complex variable z and that ψ is invertible and continuous. TDS converges in the half plane (cf. [3]) Re. $\psi(z) = \varphi(z) \leq R$. Therefore TDS represents an entire function if and only if

$$\varphi^{-1}\left[\lim_{n \to \infty} \inf. \frac{\log |a_n|}{-\lambda_n}\right] = \infty \tag{1.2}$$

Let E stands for the family of all entire Taylor Dirichlet series (ETDS - in short). The function $f(z)$, $g(z)$ and $h(z)$ are defined as

$$f(z) = \sum_{n=1}^{\infty} a_n^{\lambda_n \psi(z)}$$

$$g(z) = \sum_{n=1}^{\infty} b_n e^{\lambda_n \psi(z)}$$

and

$$h(z) = \sum_{n=1}^{\infty} c_n e^{\lambda_n \psi(z)}.$$

Throughout this paper, summation extends from 1 to ∞, unless limits are specified and f, g, h stands for $f(z)$, $g(z)$ and $h(z)$ respectively.

2. Let F be the subset of E defined by

$$F = \{f(z) : f(z) = \Sigma a_n e^{\lambda_n \psi(z)}, \ \sup_{n \geq 1} . e^{\lambda_n \varphi(n)} \, |a_n| < \infty\}$$

It can easily be seen that every element of F represents a function which converges in the whole complex plane. With pointwise addition and scalar multiplication, F becomes a linear space. The norm in F is defined as follows.

$$\|f\| = \sup_{n \geq 1} e^{\lambda_n \varphi(n)} \, |a_n|, \ f(z) = \Sigma a_n e^{\lambda_n \psi(z)} \in F.$$

It can easily be varified that F together with this norm defined on it, is a normed linear space. Multiplication in F is defined as

$$f.g = \Sigma e^{\lambda_n \varphi(n)} \, a_n b_n e^{\lambda_n \psi(z)}, \text{ for every } f, g, \in F.$$

For the definition of terms used we refer [1].

Theorem 1. F is a commutative B^*-algebra with identity.

Algebra- We just note that

$$\|f.g\| = \sup_{n \geq 1} e^{\lambda_n \varphi(n)} \, | e^{\lambda_n \varphi(n)} a_n b_n |$$

$$\leq \sup_{n \geq 1} e^{\lambda_n \varphi(n)} \, |a_n| \ \sup_{n \geq 1} . e^{\lambda_n \varphi(n)} \, |b_n|$$

$$= \|f\| \cdot \|g\|.$$

Identity. The element $e = \Sigma e_n e^{\lambda_n \psi(z)} \in F$, where $e_n = e^{-\lambda_n \varphi(n)}$ serves as identity element in F.

Completeness. Let $\{f_p\}$ be a Cauchy sequence in F, where

$$f_p(z) = \Sigma a_{pn} e^{\lambda_n \psi(z)}$$

Given $\in > 0$ we can find $p_0 \geq 1$ such that

$$\|f_p - f_q\| < \varepsilon, \text{ for every } p, q \geq p_0$$

i.e.

$$\sup_{n \geq 1} e^{\lambda_n \varphi(n)} \, |a_{pn} - a_{qn}| < \varepsilon, \text{ for every, } p\, q \geq p_0.$$

This implies that $\{a_{pn}\}$ is a Cauchy sequence in C for every n and hence, owing to the completeness of C converges to a complex number say, a_n.

Thus

$$f_p \to f, \text{ where } f(z) = \Sigma a_n \, e^{\lambda_n \psi(z)}.$$

Moreover $f(z)$ is a member of F. Since

$$e^{\lambda_n \varphi(n)} \, |a_n| \leq e^{\lambda_n \varphi(n)} \, |a_{pn} - a_n| + e^{\lambda_n \varphi(n)} |\, a_{pn}|$$

Involution. The operation * defined of F by

$$*\,(f) = f * (z) = \Sigma \, \bar{a}_n \, e^{\lambda_n \psi(z)}$$

is an involution mapping. We note that

$$\|f.\, f^* \| = \sup_{n \geq 1} .\, e^{\lambda_n \varphi(n)} \, |e^{\lambda_n \varphi(n)} \, a_n \,.\, \bar{a}_n|$$

$$= \sup_{n \geq 1} \{e^{\lambda_n \varphi(n)} \, |a_n|\}^2$$

$$= \|f\|^2.$$

Rest of the proof is straight forward.

Corollary. $(F, \|.\|)$ is a Gelfand Algebra.

Proof. Since $\|e\| = \sup_{n \geq 1} e^{\lambda_n \varphi(n)} \, |\, e^{-\lambda_n \varphi(n)}| = 1$

Theorem 2. A function $f(z) = \Sigma \, a_n \, e^{\lambda_n \psi(z)}$ in F is a topological zero divisor if and only if

(2.1) $$a_n = 0\,(e^{-\lambda_n \varphi(n)})$$

Proof. Consider the sequence $\{g_k\}$ in F, were

$$g_k(z) = e^{\lambda_k [\psi(z)} - \varphi(k)],\; k \geq 1$$

Note that for every $k \geq 1$. $g_k \in F$ and $\|g_k\| = 1$. Moreover,

$$f \cdot g_k = g_k \cdot f = e^{\lambda_k \varphi(k)} \, a_k \,.\, e^{-\lambda_k \varphi(k)} \,.\, e^{\lambda_k \psi(z)}$$

$$= a_k \,.\, e^{\lambda_k \psi(z)}.$$

Hence owing to (2.1)

(2.2) $$\|f \cdot g_k\| = \|g_k \cdot f\| \to 0 \text{ as } k \to \infty$$

Thus $f(z)$ is a topological zero divisor in F. Let on the other hand $f(z) = \Sigma a_n \, e^{\lambda_n \psi(z)}$ be a topological zero divisor in F. There exists therefore a sequence $\{g_k\}$ of elements in F with unit norm such that (2.2) holds.

Let of possible, $\lim_{n \to \infty} e^{\lambda_n \varphi(z)} |a_n| = l \neq 0$. Thus for an arbittary $\gamma > 0$, there exists n_0 such that

(2.3) $$e^{\lambda \varphi(n)} \; |a_n| > l - \gamma, \quad \text{for } n > n_0.$$

By hypothesis, for every $k > 1$

$$\sup_{n \geq 1} e^{\lambda_n \varphi(n)} \; |b_{kn}| = 1.$$

Given $\varepsilon > 0$, we can find for every k; an integer N_k and a subsequence $\{n_i\}$ of indices such that

(2.4) $$e^{\lambda_n \varphi(n)} |b_{k.n_1}| > 1 - \varepsilon \quad \text{for} \quad n_i > N_k.$$

(2.3) and (2.4) together imply that

$$\| f.g_k \| > C > 0.$$

Contradicting (2.2). Hence the theorem.

Note that F is not a division algebra. For instance, the element

$$p(z) = \Sigma \lambda_n^{-1} e^{-\lambda_n \varphi(n)} e^{\lambda_n \psi(z)}$$

though belongs to F, does not possess inverse in F. For, if possible, $q(z) = \Sigma d_n e^{\lambda_n \psi(z)}$ be its inverse. Hence we must have

$$p \, . \, q = e$$

i.e.

$$e^{\lambda_n \varphi(n)} (e^{-\lambda_n \varphi(n)} \lambda_n^{-1} d_n) = e^{-\lambda_n \varphi(n)}$$

This implies

$$d_n = \lambda_n e^{-\lambda_n \varphi(n)}.$$

It can easily be varified that the so defined $q(z) = \Sigma d_n e^{\lambda_n \psi(z)}$ is not a member of F. In this connection we propose.

Theorem-3. The function $f(z) = \Sigma a_n e^{\lambda_n \psi(z)}$ is invertible in F if and only if $\{ |e^{\lambda_n \psi(n)} a_n|^{-1} \}$ is a bounded sequence.

Proof. Let $f(z)$ be invertible and let $g(z)$ be its inverse. By hypothesis we have

$$e^{\lambda_n \psi(n)} a_n b_n \equiv e^{-\lambda_n \varphi(n)}$$

$$e^{\lambda_n \phi(n)} |b_n| = |e^{\lambda_n \phi(n)} a_n|^{-1}$$

Since $g(z) = \Sigma b_n e^{\lambda_n \varphi(z)} \in F$.We see that $\{|e^{\lambda_n \varphi(z)} a_n|^{-1}\}$ is a bounded sequence

On the other hand, if $\{|e^{\lambda_n \varphi(z)} a_n|^{-1}\}$ bounded sequence, define $g(z)$ such that

$$g(z) = \Sigma e^{-2\lambda_n \phi(n)} a_n^{-1} e^{\lambda_n \psi(z)}$$

Obviously, $g(z) \in F$. Moreover,

$$f.g = \Sigma e^{\lambda_n \varphi(n)} a_n e^{-2\lambda_n \varphi(n)} a_n^{-1} e^{\lambda_n \psi(z)}$$
$$= \Sigma e^{-\lambda_n \varphi(n)} e^{\lambda_n \varphi(z)}$$
$$= e(z).$$

Thus,

$$f.g = e.$$

In this section we consider a class G of ETDS such that

$$G = \{f : f(z) = \Sigma a_n e^{n \varphi(z)}, \Sigma e^{\lambda_n \varphi(z)} | a_n | < \infty\}$$

The norm in G is defined as

$$\|f\| = \Sigma e^{\lambda_n \varphi(z)} | a_n |.$$

With similar binary composition as in F and the norm defined in G, G becomes a normed linear space.

Theorem 4. G is a Banach* - Algebra.

Proof. We only prove the completeness.

Let $\{f_p\}$ be a Cauchy sequence in G. Given $\varepsilon > 0$, there exists some $p_0 \geq 1$, such that

$$\|f_p - f_q\| < \varepsilon, \text{ got } p, q, \geq p_0$$

(3.1) $$e^{\lambda_n \varphi(z)} |a_{pn} - a_{qn}| < \varepsilon, \text{ for } p, q \geq p_0.$$

This implies that $\{a_{pn}\}$ forms a Cauchy sequence in C for every n and hence, owing to the completeness of C, converges to a complex number, say, a_n. In (3.1), let $q \to \infty$. We get

$$e^{\lambda_n \varphi(n)} | a_{pn} - a_n | < \varepsilon, p \geq p_0$$

$$f_p \to f, \text{ where } f(z) = \Sigma a_n e^{\lambda_n \varphi(z)} \in G, \text{ since}$$

$$\Sigma e^{\lambda_n \varphi(z)} |a_n| \leq e^{\lambda_n \varphi(n)} |a_{pn} - a_n| + \Sigma e^{\lambda_n \varphi(n)} |a_{pn}|.$$

It can now be verified that G is a Banach algebra.

The proof of the theorem is complete when we define involution mapping.

$$* ! G \to C$$

as

$$* (f) = f * (z) = \Sigma\, \overline{a}_n\, e^{\lambda_n \psi(z)}$$

Note that G does not becomes a B*-algebra.

Theorem 5. Every continuous linear functional f^* on G is of the form

(3.2) $$f * (f) = \Sigma\, e^{\lambda_n \varphi(z)}\, a_n \cdot d_n, \text{ where}$$

(3.3) $\{d_n\}$ is a bounded sequence.

Conversely if (3.3) holds, then (3.2) defines a continuous linear functional on G.

Proof. We shall denote that dual space of G by G^*. Let $f^* \in G^*$. Define

$$f_n = e^{\lambda_n \{\psi(z) - \varphi(n)\}} \text{ and } f^N = \sum_{n=1}^{N} a_n\, e^{\lambda_n \psi(z)}$$

obviously, $f^N \to f$ as $N \to \infty$. Let

$$f * (f_n) = d_n.$$

Then

$$f * (f) = f * (\lim_{N \to \infty} f^N)$$

$$= f * \left(\lim_{N \to \infty} \sum_{n=1}^{N} e^{\lambda_n \varphi(z)}\, a_n f_n \right)$$

$$= \sum_{n=1}^{\infty} e^{\lambda_n \varphi(n)}\, a_n \cdot d_n$$

Moreover,

$$|d_n| = |\, f * (f_n)\, | \le M.\ \|f\| = M.$$

Conversely, let (3.3) holds. The functional defined by (3.2) is evidently well defined and linear. Further we note.

(3.4) $$\|f^*\| \le \Sigma e^{\lambda_n \varphi(n)}\, |a_n d_n| \le M\, \|f\|,$$ by (3.3).

This characterization helps us in formulating an alternative expression for the norm is G^*. We know from (2) that G is a Banach space with the same operations as in G and the norm defined as

$$\|G*\| = \sup_{|f| \le 1} |\, f * (f)\, | \,/\, \|f\|.$$

We assert that

$$\|f *\| = \sup_{n \ge 1} |d_n|.$$

By (3.4) we infer that

$$||f^*|| = \sup_{\|f\| \le 1} |f^*(f)| / \|f\| \le \sup_{n \ge 1} |d_n|.$$

on the other hand

$$|d_n| = |f^*(f_n)| \le \|f^*\| \cdot \| \|f_n\| = \|f^*\|$$

Hence the assertion.

Theorem 6. Let $f(z) = \Sigma\, a_n\, e^{\lambda_n \varphi(z)} \in G, a_n \neq 0,$ for every, $n \ge 1$. Let B be the set of complex numbers having at least one finite limit point. Define

$$f_\alpha(z) = \Sigma a_n\, e^{\lambda_n \{\psi(z) + \psi(\alpha - \varphi(n)\}}.$$

Then the set

$$S_f = (f_\alpha : \alpha \in B)$$

is a total set in G

Proof. Note first that $f_\alpha \in G$, for every $\alpha \in \beta$.

Since

$$f_\alpha(z) = \Sigma e^{\lambda_n \{\psi(z) - \varphi(n)\}}\, a_n \,.\, e^{\lambda_n \psi(z)}, \text{ and}$$

$$\Sigma\, e^{\lambda_n \varphi(n)}\, |\, a_n\, e^{\lambda_n \{\psi(\alpha) - \varphi(n)\}}\, | = \Sigma\, |a_n|\, e^{\lambda_n \varphi(\alpha)}$$

where $\varphi(\alpha) = \text{Re}\, \psi(\alpha)$,

which must converges for every $\alpha \in C$, since $f(z)$ is a function which converges absolutely in the whole complex plane.

Let $f^* \in G^*$ be such that $f^*(s_f) = 0$ *i.e.*

$$f^*(f_\alpha) = 0, \text{ for every } \alpha \in B$$

implies that $\Sigma\, e^{\lambda_n \psi(n)}\, a_n\, e^{\lambda_n \{\psi(\alpha) - \varphi(n)\}}\, d_n = 0$ for every $\alpha \in B$.

Implies that

(3.5) $$a_n\, e^{\lambda_n \psi(\alpha)}\, d_n = 0, \text{ for every } \alpha \in B.$$

Define $h(z) = \Sigma a_n\, e^{\lambda_n \psi(z)}\, d_n$. Since (3.5) holds and $f = \Sigma a_n\, e^{\lambda_n \psi(z)}$ $\in G$, $h = \Sigma a_n\, e^{\lambda_n \psi(z)}\, d_n \in G$. But owing to (3.5)

$$h(\alpha) = 0; \text{ for every } \alpha \in B.$$

Since B has a finite limit point, this means that $h \equiv 0$. This however, implies that $a_n\, d_n = 0$, for every $n \ge 1$, and as a_n is non zero for every n, we get the result.

Theorem 7. Every element of G is a topological zero divisor is G.

Proof. For the definition of topological zero divisor we refer to [1]. Consider the sequence $\{g_k\}$ where

$$g_k(z) = \Sigma\, e^{-\lambda_k \varphi(k)}\, e^{\lambda_k \psi(z)},\ k = 1, 2, 3 \ldots$$

obviously $g_k \in G$ and $\|g_k\| = 1$, and for every $k \geq 1$.

Also,

$$f \cdot g_k = g_k \cdot f = a_k \cdot e^{\lambda_k \psi(z)}$$

So,

$$\| f.g_k \| = \| g_k \cdot f \| = e^{\lambda_k \varphi(k)}\, | a_k | \to 0 \text{ as } k \to \infty.$$

Hence, the theorem.

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