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LEFT ALTERNATIVE RINGS WITH AN IDEMPOTENT

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ABSTRACT

Let R be a left alternative ring that satisfies $(x, y, z) + (y, z, x) + (z, x, y) = 0$. An idempotent element in R belongs to the nucleus of R. Let R be a prime ring with characteristic prime to 6. If R has an idempotent $e \neq 0, 1$ then R is associative. We give an example of a ring under consideration with an idempotent $e \neq 0, 1$.

1. INTRODUCTION

A non-associative ring R is called left alternative if

$$(x, y, z) + (y, x, z) = 0 \qquad \text{... (1)}$$

for all x, y, z in R where the associator $(x, y, z) = (xy)z - x(yz)$. That is,

$$(x, x, z) = 0 \qquad \text{... }(i)$$

for all x, z in R.

Kleinfed and Smith [1] have shown that a prime left alternative ring with commutators in the left nucleus and characteristic $\neq 2, 3$ is associative. Rich (2) has considered rings with idempotents in their nuclei.

2. PRELIMINARIES

In any arbitrary ring we have

$f(w, x, y, z) = (wx, y, z) - (w, xy, z) + (2, x, yz) - w(x, y, z) - (w, x, y)z = 0$

Then using (1) in $f(x, y, y, z) - f(y, y, x, z) + f(y, x, y, z) = 0$

we get $2(xy, y, z) - 2y(x, y, z) = 0$. Assuming that characteristic $\neq 2$,, we have

$$(xy, y, z) = y(x, y, z) \qquad \text{... (2)}$$

Linearizing (2) we get

$$g(x, w, y, z) = (xw, y, z) + (xy, w, z) - w(x, y, z) - y(x, w, z) = 0.$$

Using (1) and (2) in $f(x, y, y, z) = 0$, we get

$$(x^2, y, z) = (x, xy + yx, z).$$

3. MAIN SECTION

Unless otherwise stated R will be a left-alternative ring that satisfies

$(x, y, z)+(y, z, x)+(z, x, y)=0$ ... (4)

for all x, y, z in R.

Lemma 1. Let A be a non-zero ideal of R. Then $B=\{x \in R : xA=Ax=(0)\}$ is an ideal of R.

Proof. Let $a \in A, x \in B$ and $y \in R$. Then

$$0=(y, a, x)+(a, y x)=(ya)x-y(ax)+(ay)x-a(yx).$$

But $a \in A$ implies that $ya, ay \in A$. Therefore, we have $a(yx)=0$.

So $(a, y, x)=(ay)x-a(yx)=0$. Now from (4)

$$0=(y, x, a)+(x, a, y)+(a, y, x)=(y, x, a)+(x, a, y)$$
$$=(yx)a-y(xa)+(xa)y-x(ay)=(yx)a.$$

Therefore $yx \in B$. Interchanging the role of x and y in above we get $xy \in B$. Hence B is an ideal of R.

Now we assume that $((e, x), e, e)=0$ where e is an idempotent of R and $(e, x)=ex-xe$.

Theorem 2. Let R be a ring of characteristic prime to 6. If R has an idempotent $e \neq 0, 1$ then R has the desired Peirce decomposition $R=R_{11}+R_{10}+R_{01}+R_{00}$ where x belongs to R_{ij} if and only $ex=ix$ and $xe=jx$ for $i, j=0, 1$ the sum of the sub-mdodules is direct.

Proof. It suffices to show that $(e, e, x)=(e, x, e)=(x, e, e)=0$ for all x in R. Let $x \in R$. Then

$$(e, x, e)=(e^2, x, e)=(e, ex+xe, e) \quad \text{(by (3))}$$
$$=(e, ex, e)+(e, xe, e)$$

Now $0=((e, x), e, e)=(e, (e, x), e)$ (by (1))

This implies that $(e, ex, e)=(e, xe, e)$. Therefore, $(e, x, e)=2(e, ex, e)$. Therefore,

$(e, x, e)=2(e, ex, e)$. That is,

$(e, x, e)-2(e, ex, e)=0$ ... (5)

for all x in R. In particular,

$(e, ex, e)-2(e, e(ex), e)=0.$

But since R is left-alternative $(e, e, x)=0$. Therefore, $e(ex)=(ee)x=ex.$

Thus, $(e, ex, e)=0$.

Substituting in (5), we get $(e, x, e)=0$.

Now $(e, x, e)+(x, e, e)+(e, e, x)=0$ implies

that $(x, e, e)=0$. Therefore $(e, x\, e)=(x, e\, e)=(e, e, x)=0$.

Hence R has the desired Peirce decomposition.

Lemma 3. Suppose R has a Peirce decomposition with respect to e. Then

(*i*) $R_{ii}R_{ii} \subseteq R_{ii}$

(*ii*) $R_{ij}R_{kl} \subseteq \delta_{jk} R_{il}$ for all $i, j, k, l = 0, 1$.

Proof. Let $x_{11}, y_{11} \in R_{11}$. Then $e(x_{11}\, y_{11}) = -(e, x_{11}, y_{11}) + ex_{11})\, y_{11}$
$= (x_{11}, e, y_{11}) + x_{11}\, y_{11} = (x_{11}\, e) y_{11} - x_{11}(e\, y_{11}) + x_{11}\, y_{11}$
$= x_{11}\, y_{11} - x_{11}\, y_{11} + x_{11}\, y_{11} = x_{11}\, y_{11}$. And

$$\begin{aligned}(x_{11}\, y_{11})e &= x_{11}, y_{11}, e) + x_{11}(y_{11}e)\\ &= -(y_{11}, e, x_{11}) - (e, x_{11}, y_{11}) + x_{11}\, y_{11} \qquad \text{(by (4))}\\ &= -(y_{11}, e)x_{11} + y_{11}(ex_{11}) + (x_{11}, e, y_{11})\, x_{11}\, y_{11}\\ &= -y_{11}\, x_{11} + y_{11}\, x_{11} + (x_{11}e)y_{11} - x_{11}\,(ey_{11}) + x_{11}\, y_{11}\\ &= -y_{11}\, x_{11} + y_{11}\, x_{11} + x_{11}\, y_{11} - x_{11}\, y_{11} + x_{11}\, y_{11} = x_{11}\, y_{11}.\end{aligned}$$

These imply that $x_{11}\, y_{11} \in R_{11}$. Therefore $R_{11}\, R_{11} \in R_{11}$. Similarly other results can also be proved.

Nucleus : For an arbitrary non-associative ring R, the nucleus N is defined by

$$N = \{x \in R : (x, y, z) = (y, z, x) = (y, x, z) = \text{ for all } y, z \in R\}.$$

Lemma 4. Suppose R has an idempotent $e \neq 0, 1$. Then e belongs to nucleus N or R.

Proof. We shall show that $(x, e, y) = 0$ for all x, y in R. This will imply $(e, x, y) = 0$ by (1) and by (4) we get $(x, y, e) = 0$. Let $x, y \in R$ Then $R = R_{11} + R_{10} + R_{01} + R_{00}$ implies that $x = x_{11} + x_{10} + x_{01} + x_{00}$ and $y = y_{11} + y_{10} + y_{01} + y_{00}$ where $x_{ij}, y_{ij} \in R_{ij}$, $i, j = 0, 1$.

Now $(x, e, y) = (xe)y - x(ey) = (x_{11} + x_{01})y - x(y_{11} + y_{10})$
$= x_{11}\, y_{11} + x_{11}\, y_{10} + x_{01}\, y_{11} + x_{01} y_{10} - x_{11}\, y_{11} - x_{01}\, y_{11} - x_{11}\, y_{10} - x_{01}\, y_{10}$
$= 0 (\because R_{ij} R_{kl} = (0)$, if $j \neq k$, $(i, j) \neq (k, l)$ and either $i \neq j$ or $k \neq l$). Hence $e \in N$.

Lemma 5. Suppose y belongs to the nucleus N of R. Then $(y, z) \in N$ for all z in R.

Proof. Let $x, w, z \in R$ and $y \in N$. Then

$$g(x, w, y, z) = (xw, y, z) + (xy, w, z) - w(x, y, z) - y(x, w, z) = 0.$$

implies that $(xy, w, z) = y(x, w, z) = 0$. Changing x to z, w to x and z to w in the above we get

$$(zy, x, w) = y(z, x, w) \qquad \text{... (6)}$$

Now $f(y, z, x, w) = (yz, x, w) - (y, zx, w) + (y, z, xw) - y(z, x, w) - (y, z, x) = 0$. Since $y \in N$, we have

$$(yz, x, w) = (z, x, w) \qquad \text{... (7)}$$

From (6) and (7) we get $((y, z)\, x, w) = 0$. By (1) $(x, (y, z), w) = 0$ and by (4) $(x, w, (y, z)) = 0$ for all x, w in R. Hence $(y, z) \in N$.

Definition. A ring R is said to be *prime* if for any two ideals A and B of R, $AB = (0)$ implies $A = (0)$ or $B = (0)$.

Lemma 6. Let R be an arbitrary non-associative prime ring. Then R can contain no non-zero nuclear ideals (ideals in nucleus).

Proof. Let A be an ideal in the nucleus N of R. Let $x, y, z, w \in R$ and $a \in A$. Then

$$f(a, x, y, z) = (ax, y, z) - (a, xy, z) + (a, x, yz) - a(x, y, z) - (a, x, y)z = 0.$$

Since ax , $a \in A \subseteq N$, we get $a(x, y, z) = 0$.

Because $a \in A \subseteq N$, $a((x, y, z)w) = (a(x, y, z))w = 0$.

This implies that $A((R, R, R) + (R, R, R)R) = (0)$. But $(R, R, R) + (R, R, R)R$ is an ideal of R and R is prime. Either $A = (0)$ *or* $(R, R, R) + (R, R, R)R = (0)$. But R is non-associative. THerefore $= (0)$. This proves the lemma.

Lemma 7. Suppose N is the nucleus of R. Then

(*i*) $N(R, R, R) = (NR, R, R)$.

(*ii*) $(R, R, R)N = (R, R, RN)$.

(*iii*) $N(R, R, R) = (R, NR, R)$.

(*iv*) $[N, (R, R, R)] = (0)$.

Proof. Let $n \in N$ and $x, y, z \in R$. Then

(*i*) $$f(n, x, y, z) = (nx, y, z) - (n, xy, z) + (n , x, yz) - n(x, y, z) - (n , x, y)z = 0$$

implies that $(nx, y, z) = n(x, y, z)$. Therefore $(NR, R, R) = N(R, R, R)$.

(*ii*) $$f(x, y, z, n) = (xy, z, n) - (x, yz, n) + (x, y, zn) - x(y, z, n) - (x, y, z)n = 0$$

implies that $(x, y, zn) = (x, y, z)n$. Therefore $(R, R, RN) = (R, R, R)N$.

(*iii*) $(x, ny, z) = (ny, x, z) = -n(y, x, z) = n(x, y, z)$

(*iv*) $$f(x, n, y, z) = (xn, y, z) - (x, ny, z) + (x, n yz) - x(n, y, z) - (x, n, y) = 0$$

Therefore $(R, NR, R) = N(R, R, R)$.

implies that

$$(x, ny, z) = (xn, y, z) \qquad \dots (8)$$

By (*ii*)

$$\begin{aligned}
(x, y, z)n &= (x, y, zn) \\
&= -(y, zn, x) - (zn, x, y) && \text{(by (4))} \\
&= (zn, y, x) - (zn, x, y) && \text{(by (1))} \\
&= (z, ny, x) - (z, nx, y) && \text{(by (8))} \\
&= -(ny, z, x) + (nx, z, y) && \text{(by (1))} \\
&= -n(y, z, x) + n(x, z, y) && \text{(by } (i)) \\
&= n((z, y, x) + (x, z, y)) && \text{(by (1))} \\
&= -n(y, x, z) && \text{(by (4))}
\end{aligned}$$

$$= n(x, y, z) \qquad \text{(by (1))}$$

Therefore $[n, (x, y, z)] = 0$. Hence $[N, (R, R, R)] = (0)$.

Corollary 8. $[R, N] \subseteq N$.

Proof. Let $x, y, z \in R$ and $n \in N$. Then using (i), (iii) and (8), $(nx, y, z) = n(x, y, z) = (x, ny, z) = (xn, y, z)$ Therefore, $((x, n), y, z) = 0$. Using (1), $(y, (x, n), z) = 0$.

Using (4), $(y, z\,(x, n)) = 0$. Hence $(R, N) \subseteq N$.

Lemma 9. Suppose N is the nucleus of R. If I is the ideal generated by the set $[R, N)$, then $I = [R, N] + R[R, N]$.

Proof. Clearly $[R, N] + R[R, N] \subseteq I$. Let $n_1, n_2 \in N$ and $x, y, z, w \in R$. Then $[x, n_1]w = [(x, n_1), w] + w[x, n_1]$ implies that $[x, n_1]\, w \in [R, N] + R[R, N]$ since $(x, n_1) \in (R, N) \subseteq N$. Also, since $[R, N] \subseteq N$, $(y, [z, n_2])w = -y(z, n_2]w) = y[z, n_2)w] + y(w[z, n_2]) \in R[R, N] + R[R, N]$. Therefore $[R, N]$ is a right deal. Because $[R, N] \subseteq N$,

$$R(R, N] + R[R, N]) = R[R, N] + R(R[R, N)) = R[R, N] + (RR)\,[R, N]$$
$$\subseteq R[R, N] + R[R, N] \subseteq [R, N] + R[R, N].$$

Thus $[R, N] + R[R, N]$ is a two sided ideal of R. It contains $[R, N]$ which is contained in I. Hence $I = [R, N] + R[R, N]$.

Lemma 10. In the Peirce decomposition $R_{10}^2 = R_{01}^2 = (0)$.

Proof. Let $x_{10}, y_{10} \in R_{10}$ and $e \in N$. Then $(x_{10}, e, y_{10}) = 0$ implies that $(x_{10}\, e)\, y_{10} = x_{10}(ey_{10})$ or $x_{10}y_{10} = 0$. Similarly considering $(x_{01}, e, y_{01}) = 0$ we can show that $x_{10}\, y_{01} = 0$. Hence $R_{10}^2 = R_{01}^2 = (0)$.

Lemma 11. Suppose R has an idempotent $e \neq 0, 1$. Then the set $B = R_{10}\, R_{01} + R_{10} + R_{01} + R_{01}\, R_{10}$ is an ideal of R contained in the nucleus N of R.

Proof By Lemma 4, $e \in N$. By Lemma 5, $(e, x_{01}) \in N$ for some $x_{01} \in R_{01}$. This impies that $x_{01} \in N$ or $R_{01} \subseteq N$. Again $(e, x_{10}) \in N$ for some $x_{10} \in R_{10}$ implies that $R_{10} \subseteq N$. Since N is a sub- ring of R, we have both $R_{10}\, R_{01}$ and $R_{01}\, R_{10}$ contained in N. Hence $B \subseteq N$. Now using Lemma 2 and the fact that $R_{10}, R_{01} \subseteq N$, we have.

$$RB = (R_{10} + R_{01} + R_{00} + R_{11})\,(R_{10} + R_{10}\, R_{01} + R_{01}\, R_{10} + R_{01})$$
$$\subseteq R_{10}^2 + R_{10}\,(R_{10}\, R_{01}) + R_{10}\,(R_{01}\, R_{10}) + R_{10}\, R_{01} + R_{01}\, R_{10} + R_{01}\,(R_{10}\, R_{01})$$
$$+ R_{01}\,(R_{01}\, R_{10}) + R_{01}^2 + R_{00}\, R_{10} + R_{00}\,(R_{10}\, R_{01}) + R_{00}\,(R_{01}\, R_{10}) + R_{00}\, R_{01}$$
$$+ R_{11}\, R_{10} + R_{11}\,(R_{10}\, R_{01}) + R_{11}\,(R_{01}\, R_{10}) + R_{11}\, R_{01}$$
$$\subseteq O + R_{10}\, R_{11} + R_{10}\, R_{00} + R_{10}\, R_{01} + R_{01}\, R_{10} + R_{01}\, R_{11} + R_{01}\, R_{00}$$
$$+ O + O + R_{00}\, R_{11} + (R_{00}\, R_{01})\, R_{10} + R_{01} + R_{10} + (R_{11}\, R_{10})\, R_{01} + R_{11}\, R_{00} + O$$

$\subseteq O + O + R_{10} + R_{10}R_{01} + R_{01}R_{10} + R_{01} + O + O + O + O + R_{01}R_{10}$

$+ R_{01} + R_{10} + R_{10}R_{01} + O + O \subseteq B$

Similarly we can show that $BR \subseteq B$. Hence B is an ideal of R contained in N.

Theorem 12. Suppose R is a prime ring of characteristic prime to 6 has an idempotent $e \neq 0, 1$. Then R is associative.

Proof. Suppose R is not associative. Then by lemma 6, R can contain no non-zero nuclear ideal. By lemma 11, $B = (0)$. This implies that both R_{10} and R_{01} are (0) By theorem 2, $R = R_{00} + R_{11}$. Now, by lemma 3, $R_{00}(R_{00}(R_{00} + R_{11}) \subseteq R_{00}R_{00} + R_{00}R_{11} \subseteq R_{00}$ and

$(R_{00} + R_{11})R_{00} \subseteq R_{00}R_{00} + R_{11}R_{00} \subseteq R_{00}$. Therefore, R_{00} is an ideal of R. Similarly R_{11} is an ideal of R. Let $x \in R_{00}$ and $x \in R_{11}$ Then $(x, e, y) = 0$ implies $(xe)y = x(ey)$ which implies $xy = 0$. Therefore $R_{00}R_{11} = (0)$ But R is prime. Either $R_{00} =$ or $R_{11} = (0)$. But $e \neq 0 \in R_{11}$. Therefore, $R_{00} = (0)$. Then $R = R_{11}$. So $ex = xe = x$ for all x in R. Thus $e = 1$. Hence R is associative.

Example 13. Let R be a ring defined by the following multiplication table together with all finite sums of *e, a, b*

	e	a	b
e	e	0	$-b$
a	b	0	0
b	0	0	0

Note that $(e, a, e) = (ea)e - e(ae) = -eb = b$. Therefore, R is non-associative.

To prove $(x, y, z) + (y, x, z) = 0 \;\forall\; x, y, z \in R$, it suffices to check that

$(e, a, e) + (a, e, e) = 0$.

Now $(a, e, e) = (ae)e = a(ee) = be - ae = 0 - b = -b$

$\therefore\; (e, a, e) + (a, e, e) = b - b = 0$.

Also $(e, e, a) = (ee)a - e(ea) = ea = 0$.

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