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LAURICELLA'S FUNCTIONS AND THEIR TRANSFORMATIONS. I

By

S.C. Gupta and B.M. Agarwal

33-C Jawahar Nagar, Lashkar, Gwalior - 474001, M.P., India

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ABSTRACT

In this paper we shall obtain a transformation for Lauricella's function with the help of difference operators.

1. INTRODUCTION.

It is well known that difference operators Δ and E play a very impotant role in the field of special functions. Agarwal [1, 2, 3] and Gupta and Agarwal [5, 6, 7] have successfully applied these operators to obtain various tranformations and also evaluated several integrals in a very simple manner. In this paper we shall obtain the transformation [2, p. 116]

$$F_A[a; b_1, b_2, ..., b_n; c_1, c_2, ..., c_n; x_1, x_2, ..., x_n]$$

$$= (1 - x_1 - ... - x_k)^{-a} F_A[a; c_1 - b_1, c_2 - b_2, ..., c_k - b_k, b_{k+1},$$

$$b_{k+2}, ..., b_n; c_1, ..., c_n ; \frac{-x_1}{1 - x_1 - ... - x_k}, ...,$$

$$\left.\frac{-x_k}{1 - x_1 - ... - x_k}, \frac{x_{k+1}}{l - x_1 - ... - x_k}, ..., \frac{x_n}{1 - x_1 - ... - x_k}\right] \quad (*)$$

for Lauricella's functions with the help of difference operators.

2. FIRST STEP OF PROOF

We know that

$$F_2(a; b_1, b_2; c_1, c_2; x_1, x_2]$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(a)_{m_1+m_2} (b_1)_{m_1} (b_2)_{m_2} x_1^{m_1} x_2^{m_2}}{(c_1)_{m_1} (c_2)_{m_2} m_1! \, m_2!}$$

$$= \frac{\Gamma(c_1)\,\Gamma(c_2)}{\Gamma(b_1)\;\Gamma(b_2)} (1 - x_1E_1 - x_2E_2)^{-a} \frac{\Gamma(b_1)\,\Gamma(b_2)}{\Gamma(c_1)\;\Gamma(c_2)},$$

where $E_1^n f(\alpha_1) = f(\alpha_1 + n)$ and E_i operates on b_i and c_i only such that

$$\frac{\Gamma(c_i)}{\Gamma(b_i)} E_i^n \left(\frac{\Gamma(b_i)}{\Gamma(c_i)}\right) = \frac{(b_i)_n}{(c_i)_n} \quad \ldots. \; (**)$$

This suggests a generalization of this method i.e., consider

$$A = \left[\frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_1) \ldots \Gamma(b_n)}\right] . (1 - x_1 E_1 - \ldots - x_n E_n)^{-a} \left[\frac{\Gamma(b_1) \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)}\right]. \quad \ldots (1)$$

Now we shall show with the help of following Lemmas that this general form reduces to Lauricella's function F_A.

Lemma 1. It is well known that (multinomial Theorem)

$$(x_1 + \ldots + x_n)^k = \sum_{\Sigma m_i = k} \frac{k!\, x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!}.$$

Lemma 2 (Srivastava. [8, p. 4]).

$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} (x_1 + \ldots + x_n)^k$$

$$= \sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n}}{m_1! \ldots m_n!} x_1^{m_1} \ldots x_n^{m_n}.$$

It can be easily proved either by induction method [8, p. 4] or with the help of Lemma 1.

Now consider

$$A = \left[\frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_1) \ldots \Gamma(b_n)}\right]$$

$$.(1 - x_1 E_1 - \ldots - x_n E_n)^{-a} \left[\frac{\Gamma(b_1, \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)}\right],$$

$$= \left[\frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_1) \ldots \Gamma(b_n)}\right] \sum_{k=0}^{\infty} \frac{(a)_k}{k!} .(x_1 E_1 + \ldots + x_n E_n)^k \left[\frac{\Gamma(b_1) \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)}\right].$$

In view of Lemma 1 and Lemma 2 we have,

$$A = \left[\frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_1) \ldots \Gamma(b_n)}\right] \sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n}}{m_1! \ldots m_n!}$$

$$. (x_1 E_1)^{m_1} \ldots (x_n E_n)^{m_n} \left[\frac{\Gamma(b_1) \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)}\right]$$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n}}{m_1! \dots m_n!} \cdot \frac{(b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \cdot x_1^{m_1} \dots x_n^{m_n}.$$

$$= F_A\,[a; b_1, \dots, b_n; c_1, \dots, c_n; x_1, \dots, x_n], \qquad \dots (2)$$

where, for convergence,

$$|x_1| + x_2| + \dots + |x_n| < 1.$$

It has now been shown that A equals the 1hs of (*).

3. SECOND STEP

To prove that A equals also the rhs of (*) we set $E_i = 1 + \Delta_i$. It can be shown that

$$\frac{\Gamma(c_i)}{\Gamma(b_i)} \Delta_i^m \left(\frac{\Gamma(b_i)}{\Gamma(c_i)}\right) = \frac{(-1)^m (c_i - b_i)_m}{(c_i)_m}. \qquad \dots (***)$$

Thus,

$$A = \left[\frac{\Gamma(c_1) \dots \Gamma(c_n)}{\Gamma(b_1) \dots \Gamma(b_n)}\right] (1 - x_1E_1 - \dots - x_nE_n)^{-a} \left(\frac{\Gamma(b_1) \dots \Gamma(b_n)}{\Gamma(c_1) \dots \Gamma(c_n)}\right)$$

$$= \left[\frac{\Gamma(c_1) \dots \Gamma(c_n)}{\Gamma(b_1) \dots \Gamma(b_n)}\right] (1 - x_1 - \dots - x_k)^{-a}$$

$$\cdot \left[1 - \frac{x_1\Delta_1 + \dots + x_k\Delta_k + x_{k+1}E_{k+1} + \dots + x_nE_n}{1 - x_1 - \dots x_k}\right]^{-a} \left(\frac{\Gamma(b_1) \dots \Gamma(b_n)}{\Gamma(c_1) \dots \Gamma(c_n)}\right)$$

Again using Lemma 1 and Lemma 2 we have,

$$A = \left[\frac{\Gamma(c_1) \dots \Gamma(c_n)}{\Gamma(b_1) \dots \Gamma(b_n)}\right] (1 - x_1 - \dots x_k)^{-a}$$

$$\cdot \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n}}{m_1! \dots m_n!} \left(\frac{x_1\Delta_1}{1 - x_1 - \dots - x_k}\right)^{m_1}$$

$$\cdots \left(\frac{x_k\Delta_k}{1 - x_1 \dots - x_k}\right)^{m_k} \left(\frac{x_{k+1}E_{k+1}}{1 - x_1 - \dots - x_k}\right)^{m_{k+1}} \cdots$$

$$\left(\frac{x_nE_n}{1 - x_n - \dots - x_k}\right)^{m_n} \left(\frac{\Gamma(b_1) \dots \Gamma(b_n)}{\Gamma(c_1) \dots \Gamma(c_n)}\right),$$

Hence, from (1), (2) and (***), we see that A also equals the rhs of (*).

This completes the proof.

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REFERENCES

[1] B.M. Agarwal, On multiple sum, *La Ricerca* (Napoli) (1968), 20-24

[2] B.M.Agrawal, Transformations of hypergeometric series, *Vijnana Parishad Anusandhan Patrika*, **16** (1973), 169-176.

[3] B.M. Agrawal, Application of Δ and E operators to evaluate certain integrals, *Proc. Camb. Phil. Soc.* (1968), 89-104.

[4] P. Appell and J. Kampe de Feriet, *Functions hyper-geometriques et hyperspheriques*, Paris, Gauthier-Villars, 1926.

[5] S.C. Gupta and B.M. Agrawal, Finite summation, I, *J. Indian Acad. Math.* **14** (1992), 80-85.

[6] S.C. Gupta and B.M. Agrawal, Finite summation. II, *J. Indian Acad. Math* **14** (1992), 89-93.

[7] S.C. Gupta and B.M. Agrawal, Finite summation III. *J. Indian Acad Math.* **17** (1995), 87-101.

[8] H.M. Srivastava, Certain double integrals involving hypergeometric functions, *Jñānābha*, Sect. A, 1 (1971), 1-10.