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ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BELONGING TO THE LIPSCHITZ CLASS BY $F(a, q)$ MEANS

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ABSTRACT

In the present paper, we obtain the degree of approximation of $f \in \text{Lip}\,\alpha$ $(0 < \alpha \leq 1)$ by $F(a, q)$ means of its Fourier series.

1. INTRODUCTION

Let $C_{2\pi}$ be the space of all 2π - periodic and continuous functions defined on $[-\pi, +\pi]$, which is Banach space under "Sup" norm. For each $f \in C_{2\pi}$, let the Fourier series be given by

$$(1.1) \qquad f(x) \sim \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x).$$

We write

$$(1.2) \qquad \Phi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}.$$

The family $F(a, q)$ of summability methods was introduced by Meir [5]. The summability matrix $\{C_{pk}\}$ belongs to $F(a, q)$ if it satisfies the following conditions :

p is a discrete or continuous parameter : $q = q(p)$ is a positive increasing function which tends to infinity as $p \to \infty$; 'a' is a positive constant : for every fixed $\delta : 1/2 < \delta < 2/3$

$$(1.3) \qquad C_{pk} = \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$

$$\{1 + 0\left(\frac{|k-q|+1}{q}\right) + 0\left(-\frac{|k-q|^3}{q^2}\right)\}$$

as $p \to \infty$ uniformly in k for $|k - q| \leq q^{\delta}$; and

$$(1.4) \qquad \sum_{|k-q| > q^{\delta}} (k+1)\, C_{pk} = 0\,\{\exp(-q^{\mu})\},$$

where μ is some positive number independent of p.

Let

(1.5) $$t_p(f:x)=\sum_{k=0}^{\infty} C_{pk}\, s_k(f:x)$$

denote the $F(a, q)$ mean of the Fourier series (1.1) of f, where $s_k(f:x)$ is is the kth partial sum of (1.1).

The family $F(a, q)$ contains the summability methods of generalised Borel, Euler, Taylor, S_α and (e, c).

It is known (see [4]) that

(1.6) $$\sum_{k=0}^{\infty} C_{pk} = 1 + 0\,(q^{-1/2}).$$

The summability methods of Euler, Taylor, S_α and Borel satisfy (1.6) in the stronger form

(1.7) $$\sum_{k=0}^{\infty} C_{pk} = 1.$$

2. Degree of approximation by borel means and (E, q) - means were obtained by Chandra [1] and [2] respectively. Extending the results of Chandra to (e, c) mean we [6] have proved the following theorems :

THEOREM A. Let $f \in C_{2\pi} \cap \text{Lip}\,\alpha$, $0 < \alpha \le 1$. Then

(2.1) $$\|t_n^c - f\| = 0\,(n^{-\alpha/2}),$$

where $t_n^c\,(f; x)$ is n^{th} (e;, c) means of the Fourier series of f at x.

Since $F(a, q)$ method includes (e, c) method, it is natural to ask as to what will be the result if we apply $F(a, q)$ mean to obtain the degree of approximation for $f \in \text{Lip}\,\alpha\ (0 < \alpha \le 1)$?

We shall prove the following theorem :

THEOREM : Let $[q]$ denote the integral part of $q = q(p)$ and $m = [q] + 1$. If $f \in \text{Lip}\,\alpha$, $0 < \alpha \le 1$, then

(2.2) $$\|t_p(f\,; x) - f(x)\| = \sup_{-\pi \le x \le \pi} |t_p(f:x) - f(x)| = 0(m^{-\alpha/2}).$$

3. For the proof of our theorem we shall need the following lemmas :

LEMMA 1. If $q = q(p)$, is an integer valued function of p, then, for $\frac{1}{2} < 6 < \frac{2}{3}$ we have

$$\int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}}\, \exp\,(-aq^{-1}(k-q)^2)\, \sin\,(k+1/2)t\, dt$$

$$= \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \exp\left(-\frac{qt^2}{4a}\right) \sin\left(q+\frac{1}{2}\right)t\, dt + O(q \exp\,(-aq^{2\delta-1})).$$

PROOF OF LEMMA 1. From the proof of Lemma 3.2 of Ikeno [3] we have

$$\sum_{|k-q|\le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t$$
$$= \exp(-\frac{qt^2}{4a}) \sin(q+\frac{1}{2})t\, dt + O(q \exp(-aq^{2\delta-1})\ |t|).$$

Hence the Lemma 1 follows in view of boundedness of $\Phi_x(t)$ and since $\sin(t/2) > t/\pi$ $(0 < t < \pi)$.

LEMMA 2. If m is as defined in the theorem, we have

$$\int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{|k-q|\le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t\, dt$$
$$= \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{|k-m|\le m^{6}} \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \sin(k+1/2)t\, dt$$
$$+ O(m^{-\alpha})$$

PROOF OF LEMMA 2. We estimate the difference :

$$\int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{|k-q|\le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t\, dt$$
$$- \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{|k-m|\le m^{\delta}} \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \sin(k+1/2)t\, dt$$
$$= \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{m\le k\le m+m^{\delta}} \left[\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\right.$$
$$\left. - \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m^2))\right] \sin(k+1/2)t\, dt$$
$$+ \int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \sum_{m-m^{\delta}\le k<m} \left[\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\right.$$
$$\left. - \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m^2))\right] \sin(k+1/2)t\, dt$$
$$- \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{q+q^{\delta}<k\le m+m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t\, dt$$
$$+ \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{q-q^{\delta}\le k\le m-m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$

$$\sin (k + 1/2)t\, dt$$

(3.1) $= D_1 + D_2 + D_3 + D_4.$

From Ikeno [3], p. 261, 262, we have

$$\sum_{q+q^\delta < k \le m+m^\delta} \sqrt{\frac{a}{\pi q}} \exp (-aq^{-1}(k-q)^2).\sin (k+1/2)t\, dt$$
$$= O(\sqrt{q}\, t \exp (-aq^{2\delta-1}))$$

and also

$$\sum_{q-q^\delta \le k \le m-m^\delta} \sqrt{\frac{a}{\pi q}} \exp (-aq^{-1}(k-q)^2)\sin (k+1/2)t\, dt$$
$$= O(\sqrt{q}\, t \exp (-aq^{2\delta-1}))$$

Thus,

$$|D_3| \le \int_0^\pi \frac{\|\Phi_x(t)\|}{t} \sum_{q+q^\delta < k \le m+m^\delta} \sqrt{\frac{a}{\pi q}} \exp (-aq^{-1}(k-q)^2)\, |\sin (k+1/2)t|\, dt$$

(3.2) $\qquad = O(\sqrt{q} \exp (-aq^{2\delta-1})).$

Similarly,

(3.3) $\qquad |D_4| = 0(\sqrt{q} \exp (-aq^{2\delta-1})).$

In case where $q < m \le k \le m + m^\delta$, we have

$$0 \le \frac{(k-m)}{\sqrt{m}} < \frac{(k-q)}{\sqrt{q}} < \frac{(k-[q])}{\sqrt{[q]}}.$$

Therefore, from Ikeno [3], p. 259

(3.4) $$\left| \sqrt{\frac{a}{\pi q}} \exp (-aq^{-1}(k-q)^2) - \sqrt{\frac{a}{\pi m}} \exp (-am^{-1}(k-m)^2) \right|$$
$$= O\{\frac{1}{\sqrt{m}} \exp (-am^{-1}(k-m)^2) (\frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m})\}.$$

Using (3.4), we obtain

$$|D_1| = O\ [\int_0^\pi \frac{|\Phi_x(t)|}{t} \sum_{m \le k \le m+m^\delta} \{\frac{1}{\sqrt{m}} \exp (-am^{-1}(k-m)^2)$$
$$(\frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m})\}.\ |\sin (k+1/2)t|\ dt]$$

$$= 0\ [\int_0^{\pi/m} \frac{|\Phi_x(t)|}{t} \sum_{m \le k \le m+m^\delta} \{\frac{1}{\sqrt{m}} \exp (-am^{-1}(k-m)^2)$$

$$(\frac{(k-m)^2}{m^2}+\frac{|k-m|}{m}+\frac{1}{m})\}. (|k-m| + m + 1/2)t\, dt$$

$$+ O\ [\int_{\pi/m}^{\pi} \frac{|\phi_x(t)|}{t} \sum_{m \le k \le m+m^{\delta}} \{\frac{1}{\sqrt{m}} \exp(-am^{-1}(k-m)^2)$$

$$(\frac{(k-m)^2}{m^2}+\frac{|k-m|}{m}+\frac{1}{m})\}dt]$$

$$= O\{\sqrt{m}\int_0^{\pi/m} |\phi_x(t)|\, dt\} + 0\{\frac{1}{\sqrt{m}}\int_{\pi/m}^{\pi} \frac{|\phi_x(t)|}{t}\, dt\}$$

(3.5) $\quad = O(m^{-\alpha})$.

Also in case where $k \le [q] < q < m$, we have

$$\frac{(k-m)}{\sqrt{m}} < \frac{(k-q)}{\sqrt{q}} < \frac{(k-[q])}{\sqrt{[q]}} < 0.$$

Hence, from Ikeno [3, p. 260] following estimate results :

$$|\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) - \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2)|$$

$$= O\{\frac{1}{\sqrt{[q]}} \exp(-a[q]^{-1}(k-[q])^2)(\frac{(k-[q])^2}{[q]^2}+\frac{|k-[q]|}{[q]}+\frac{1}{[q]})).$$

Proceeding as in the estimation of D_1 and using the above inequality we get

(3.6) $\quad |D_2| = O(m^{-\alpha})$.

Thus, Lemma 2 follows from (3.1) to (3.6).

4. PROOF OF THE THEOREM.

Since $S_k(f;x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \sin(k+1/2)t\, dt$

we have

$$t_p(f;x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \sum_{k=0}^{\infty} C_{pk} \sin(k+1/2)t\, dt + O(q^{-1/2})$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} [(\sum_{|k-q| \le q^{\delta}} + \sum_{|k-q| \le q^{\delta}})C_{pk} \sin(k+1/2)t]dt + O(q^{-1/2})$$

(4.1) $\quad = S_1 + S_2 + 0(q^{-1/2})$.

By (1.4) and the fact that $\sin t/2 \ge t/\pi$ $(0 < t \le \pi)$, we have

$$|S_2| \le \frac{|\phi_x(t)|}{t} \sum_{|k-q| > q^{\delta}} C_{pk}(k+1/2)t\, dt$$

(4.2) $\quad = O(\exp(-q^{\mu}))$.

Now making use of (1.3), we can write

$$S_1 = \frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$
$$\{1 + O(\frac{|k-q|+1}{q}) + O(\frac{|k-q|^3}{q^2})\}. \sin(k+1/2)t\, dt$$

(4.3) $\quad = S_3 + S_4 + S_5$.

We estimate S_4 as follows :

$$|S_4| \le \int_0^{\pi} \frac{|\phi_x(t)|}{t} \sum_{|k-q| \le q^{\delta}} \{\sqrt{\frac{a}{\pi q}} \exp - (aq^{-1}(k-q)^2)$$
$$O(\frac{|k-q|+1}{q})\} \, |\sin(k+1/2)t|\, dt$$

$$= \int_0^{\pi/q} \frac{|\phi_x(t)|}{t} \sum_{|k-q| \le q^{\delta}} \{\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$
$$O(\frac{|k-q|+1}{q})\} \, (|k-q| + q + 1/2)t\, dt$$

$$+ \int_{\pi/q}^{\pi} \frac{|\phi_x(t)|}{t} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$
$$O(\frac{|k-q|+1}{q})\, dt$$

$$= O(\sqrt{q}\int_0^{\pi/q} t^{\alpha}\, dt)\, O(\frac{1}{\sqrt{q}} \int_{\pi/q}^{\pi} t^{\alpha-1}\, dt)$$

(4.4) $\quad = O(q^{-\alpha})$.

Similarly,

(4.5) $\quad |S_5| = O(q^{-\alpha})$.

Applying Lemma 1 and Lemma 2 and noting that $m = m(p)$ is an integer valued function of p we get

$$S_3 = \int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \exp(-\frac{mt^2}{4a}) \sin(m + 1/2)t\, dt.$$

Now

$$|S_3| = O(1)\int_0^{\pi} t^{\alpha-1} \exp(-\frac{mt^2}{4a})\, |\sin(m+1/2)t|\, dt$$

$$= O(1) \int_{0}^{\pi/\sqrt{m}} t^{\alpha-1}\, dt + O(m^{-1}) \int_{\pi/\sqrt{m}}^{\pi} t^{\alpha-2} \frac{\partial}{\partial t} (\exp(-\frac{mt^2}{4a}))dt$$

(4.6) $= O(m^{-\alpha/2})$.

Collection of (4.1), (4.2) , ..., (4.6) completes the proof of the Theorem.

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