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COMMON FIXED POINT OF TWO PAIRS OF COMMUTING MAPPING IN SAKES SPACE

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ABSTRACT

Meir and Keeler [1] obtained a remarkable generalisation of Banach contraction principle. In this paper, a common fixed point theorem for the pairs of commuting mapping in a Saks space, satisfying Meir and Keeler type condition is obtained.

Theorem Let P and S be commuting mappings and Q and T be commuting mappings of a Saks space $(Xs.d) = (X.N_1 . N_2)$ into itself satisfying the following conditions :

Given $\in > 0$, there exists a $\delta(\in) > 0$, ($\delta(\in)$ being a non decreasing function of $\in$) such that for all $x, y \in X$, $\in \leq K \max\{N_2(Sx - Ty),$
$N_2(Px - Sx), N_2(Qy - Ty), N_2(Px - Ty), \quad N_2(Sx - Qy)\} < \in + \delta$

$$\Rightarrow N_2(Px - Qy) < \in \qquad \text{... (1)}$$

$$Px = Qy \text{ whenever } Px = Sx, Qy = Ty \qquad \text{... (2)}$$

where $0 \leq 2K < 1$.

Further if the range of T contains the range of P and the range of S contains the range of Q and if one of P, Q, S and T is continuous then P, Q, S and T have a common fixed point z. Further z is the unique common fixed point of P and S of Q and T.

Proof : For the proof of this theorem we recall the lemma of Orlicz [2] for a Saks space.

Lemma Let $(Xs, d) = (X, N_1, N_2)$ be a Saks space. Then the following statements are equivalent.

(*i*) N_1 is equivalent to N_2 on X

(*ii*) (X, N_1) is a Banach space and $N_1 \leq N_2$ on X

(*iii*) (X, N_2) is a Frechet space and $N_2 \leq N_1$ on X.

Now we shall prove the theorem. First with the help of (1), we note that for all x, y in X such that

$$Px \neq Sx, Qy \neq Ty, N_2(Px - Qy) < K \max\{N_2(Sx - Ty), N_2(Px - Sx)$$
$$N_2(Qy - Ty), N_2(Px - Ty), N_2(Sx - Qy)\}.$$

Next the non decreasing character of $\delta(\in)$ implies that given $\in > 0$, there exists $\in > 0$ such that

$$\in' < \in < \in' + \delta(\in') \text{ or equivalently}$$
$$K \max\{N_2(Sx - Ty), N_2(Px - Sx), N_2(Qy - Ty), N_2(Px - Ty), N_2(Sx - Qy)\} = \in \quad \Rightarrow N_2(Px - Qy) < \in', \in' < \in \quad \text{... (4)}$$

Let x_0 be an arbitrary point in X Choose a point x_1 and then a point x_2 in X such that $Px_0 = Tx_1$ and $Qx_1 = Sx_2$.

This can be done as $PX \subset TX$. In general having chosen the point x_{2n} choose a point x_{2n+1} and then a point x_{2n+2} such that

$$Px_{2n} = Tx_{2n+1} \text{ and } Qx_{2n+1} = Sx_{2n+2}.$$

Then we have,

$$N_2(Px_{2n} - Qx_{2n+1}) < K \max\{N_2(Sx_{2n} - Tx_{2n+1}), N_2(Px_{2n} - Sx_{2n}), N_2(Qx_{2n+1} - Tx_{2n+1}), N_2(Px_{2n} - Tx_{2n+1}, N_2(Sx_{2n} - Qx_{2n+1})\},.$$

$$N_2(Px_{2n} - Qx_{2n+1}) < K \max\{N_2(Qx_{2n-1} - Px_{2n}), N_2(Px_{2n} - Qx_{2n-1}), N_2(Qx_{2n+1} - Px_{2n}), N_2(Px_{2n} - Px_{2n}), N_2(Sx_{2n} - Qx_{2n+1})\}$$

$$N_2(Px_{2n} - Qx_{2n+1}) < K \max\{N_2(Qx_{2n-1} - Px_{2n}), N_2(Qx_{2n-1} - Qx_{2n+1})\}$$
$$= K \max\{N_2(Qx_{2n-1} - Px_{2n}), N_2(Qx_{2n-1} - Px_{2n}) + N_2(Px_{2n} - Qx_{2n+1})\}.$$

Or $N_2(Px_{2n} - Qx_{2n+1}) < K[N_2(Qx_{2n-1} - Px_{2n}) + N_2(Px_{2n} - Qx_{2n+1}\}$,

i.e. $\quad N_2(Px_{2n} - Qx_{2n+1}) < K/(1-K) N_2(Qx_{2n-1} - Px_{2n})$.

Similarly we have

$$N_2(Qx_{2n-1} - Px_{2n}) = N_2(Px_{2n} - Qx_{2n-1}) < K/(1-K) N_2(Qx_{2n-1} - Px_{2n-2}).$$

From the last two inequalities we conclude that both $N_2(Px_{2n} - Qx_{2n+1})$ and $N_2(Qx_{2n+1} - Px_{2n+2}) = 0$ are monotonic decreasing sequences of positive real numbers.

Now $\lim_{n \to \infty} N_2(Px_{2n} - Qx_{2n+1}) = 0$

and $\lim_{n \to \infty} N_2(Qx_{2n+1} - Px_{2n+2}) = 0$.

For, if not, suppose for instance

$\lim_{n \to \infty} N_2(Px_{2n} - Qx_{2n+1}) = r$ where $r > 0$.

Then given $\delta > 0$ there exists a positive integer N such that for each integer $m > N$, we heave

$$r \le N_2(Px_{2m} - Qx_{2m+1}) < r + \delta \quad \text{... (5)}$$

Or $r \le K \max\{N_2(Sx_{2m+2} - Tx_{2m+1}), N_2(Px_{2m+2} - Sx_{2m+2}),$

$$N_2(Qx_{2m+1}) - Tx_{2m+1}),$$
$$N_2(Px_{2m+2} - Tx_{2m+1}), N_2(Sx_{2m} - Qx_{2m+1})\} < r + \delta \quad \ldots (6)$$

Selecting δ in (6) in accordance with (1), for each $m \geq N$ we obtain $N_2(Px_{2m} - Qx_{2m+1}) < r$ and so, $N_2(Px_{2m+2} - Qx_{2m+3}) < r$ which contradicts (5). Therefore we have

$$\lim_{n \to \infty} N_2(Px_{2n} - Qx_{2n+1}) = 0$$

and similarly, $\lim_{n \to \infty} N_2(Qx_{2n+1} - Px_{2n+2}) = 0.$

Now by lemma,

$\lim_{n \to \infty} N_1(Px_{2n} - Qx_{2n+1}) = 0 = \lim_{n \to \infty} N_1(Qx_{2n+1} - Px_{2n+2})$.From these limits and by condition (1) it follows easily that $\{Px_0, Qx_1, Px_2, Qx_3, \ldots, Px_{2n}, Qx_{2n+1}, \ldots)$ is a Cauchy sequence in the Saks space X and so has a limit z in X.

Hence the sub-sequence $\{Px_{2n}\} = \{Tx_{2n+1})$ and $\{Qx_{2n+1})$ $= \{Sx_{2n+2}\}$ converge to the point z.

Let us now suppose that the mapping S is continuous. Then since the mappings P and S commute, the sequences $\{Sx_{2n}\}$ and $\{PSx_{2n}\}$ converge to the point Sz. Now we shall show that $Sz = z$. For if $z \neq Sz$, then $N_2(PSx_{2n} - Qx_{2n+1} < K \max\{S^2x_{2n} - Tx_{2n+1}),$
$N_2(PSx_{2n} - S^2x_{2n}), N_2(Qx_{2n+1} - Tx_{2n+1}), N_2(PSx_{2n} - Tx_{2n+1}),$

$$N_2(S^2x_{2n} - Qx_{\prime}2n + 1))\}.$$

Letting $n \to \infty$ on both sides we have

$N_2(Sz - z) \leq K N_2(Sz - z)$ which is a contradiction. Hence we have $Sz = z$. Similarly,

$$N_2(Pz - Qx_{2n+1}) < K \max\{N_2(Sz - Tx_{2n+1}), N_2(Pz - Sz),$$
$$N_2(Qx_{2n+1} - Tx_{2n+1}), N_2(Pz - Tx_{2n+1}), N_2(Sz - Qx_{2n+1})\}.$$

Letting $n \to \infty$ on both sides, we get $z = Pz$.

Thus we have $z = Pz = Sz$. It shows that there exists a point z' in X such that $Sz = z = Pz = Tz'$, since the range of P is contained in the range of T.

Now, $QTz' = Qz = TQz'$ ... (A) as Q and T commute. Further $z = Qz'$ for

$$N_2(z - Qz') = N_2(Pz - Qz') < K \max\{N_2(Sz - Tz'), N_2(Pz - Sz),$$
$$N_2(Qz' - Tz'), N_2(Pz - Tz'), N_2(Sz - Qz')\} = K N_2(z - Qz'),$$

which implies $z = Qz'$, By (A) and $z = Qz'$, $Qz = Tz$. Thus $z = Qz = Tz$ for otherwise

$$N_2(z - Qz) = N_2(Pz - Qz) < K \max \{N_2(Sz - Tz), N_2(Pz - Sz), N_2(Qz - Tz)\, N_2(Pz - Tz), N_2(Sz - Qz)\} = K\, N_2(z - Qz) \text{ which implies } z = Qz.$$

Thus we have $z = Pz = Sz = Qz = Tz$. So far we have proved that z is a common fixed point of P, Q, S and T. The proof is similar if T is continuous instead of S.

Now suppose that the mapping is continuous. Since P and S commute the sequences $\{P^2x_{2n}\}$ and $\{SPx_{2n}\}$ converge to Pz. So,

$$N_2(P^2x_{2n} - Qx_{2n+1}) < K \max \{N_2(SPx_{2n} - Tx_{2n+1}), N_2(P^2x_{2n} - SPx_{2n}), N_2\, Qx_{2n+1} - Tx_{2m+1}, N_2(P^2x_{2n} - T\,x_{2n+1}), N_2(SPx_{2n} - Qx_{2n+1})\}$$

on letting $n \to \infty$ on both sides will lead to $z = Pz$. This shows there exists a point z' in X such that $z = Pz = Tz'$ as we have $z \in PX$ and since $PX \subset TX$, Then we have,

$$N_2(P^2x_{2n} - Qz') < K \max \{N_2\,(Sx_{2n} - Tz'), N_2(P^2x_{2n} - SPx_{2n}), N_2(Qz' - Tz'), N_2(P^2x_{2n} - Tz'), N_2(SPx_{2n} - Q(z')\}$$

on letting $n \to \infty$ on both sides, we obtain $z = Qz'$

i.e. $Tz = TQz' = QTz' = Qz$. Further

$$N_2(Px_{2n} - Qz) < K \max \{N_2\,(Sx_{2n} - Tz), N_2(Sx_{2n} - Px_{2n}), N_2(Qz - Tz), N_2(Px_{2n} - Tz), N_2(Sx_{2n} - Qz)\}$$

Again on letting $n \to \infty$ we get, $z = Qz = Tz$. This shows that there exists a point z'' in X such that $z = Qz = Sz''$ with the same reasoning that $QX \subset SX$. Now,

$$N_2(PZ'' - z) = N_2(Pz'' - Qz) < K \max \{N_2(Sz' - Tz), N_2(Pz'' - Sz''), N_2(Qz - Tz), N_2(Pz'' - Pz), N_2(Sz'' - Qz)\}.$$

From which again we get $z = Pz'' = Sz''$. Since P and S commute $PSz'' = Sz = SPz'' = Pz = z$. Thus we have proved that z is again a common fixed point of P, Q, S and T. If the mapping Q is continuous instead of P, then the proof that z is again a comon fixed point of P, Q, S and T is similar.

Now we shall show the uniqueness of the point z. Suppose that w is also a common fixed point of P and S. Then,

$$N_2(w - z) = N_2(Pw - Qz) < K \max \{Sw - Tz), N_2(Pw - Sw), N_2(Qz - Tz), N_2(Pw - Tz)\} = K\, N_2(w - z)$$

which leads us to conclude $w = z$. Thus z is the unique common fixed point of P and S. Similarly we can prove that z is the unique common fixed point of Q and T.

REFERENCES

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[2] W. Orlicz, Linear Operations in Saks Space (I), *Stud. Math,* **11** (1950), 237.