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SOME COUNTEREXAMPLES IN FIXED POINT THEORY

by

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ABSTRACT

We give a few counterexamples in Fixed Point Theory for multivalued mappings.

In this short paper we present a few counterexamples of various nature in the area of Fixed Point Theory for multivalued mappings.

They are not directed to an audience of experts in the field, but they are devoted to an audience of neophytes of Fixed Point Theory; neophytes are above all rich of little experience.

Although the counterexamples presented here are very elementary, we think they can be more useful than laborious but "fancy" fixed point Theorems.

In the following we will label :

– H a Hilbert space.

– E a Banach space with norm $\| \ \|$.

– X a metric space with distance d.

– 2^X the non empty subsets of X.

– K a subset of X.

– $CB(X)$ the family of nonempty bounded closed subsets of X.

– $KC(E)$ the family of nonempty compact convex subsets of E.

– For $A, B \in CB(X)$, $D(A, B) := \max\,(\sup_{b \in B} d(b, A), (\sup_{b \in A} d(a, B))$,

where $d(X, Y) := \inf_{y \in Y} d(x, y)$, is the Hausdorff distance from A to B.

– f a single-value map from K to X.

– T a multivalued map from K to 2^X.

– $Fix(T)$ the set of fixed points of T, $Fix(T) := \{z \in K : z \in Tz\}$.

– For $A, B \in CB(X)$, $\delta(A, B) := \sup \{d(a, b) :$
$a \in A, b \in B\}$, $\delta(x, A) := \delta(\{x\}, A)$.

COUNTEREXAMPLES 1, 2 AND 3.

In [1] Caristi states (and it could not be said better): "An inwardness condition is one which assert that, in some sense, points from the domain are mapped "toward" the domain . . . Possibly the weakest of the inwardness conditions, the Leray- Schauder boundary condition, is the assumption that T maps points x of ∂K anywhere except to the outward part of the ray originating at some interior point of K and passing through x, i.e.

$\exists\, w \in K^0$ such that " $x \in \partial K,\ f(x) - w = m(x - w)$ implies $m \leq 1$. ... (1)

If $x \in K$ we define the inward set, $I_K(x)$, as follows :

$$I_K(x) := \{x + c(u - x) \in E : u \in K \text{ and } c \geq 1\}.$$

A mapping $f : K \to E$ is said to be inward if $f(x) \in I_K(x) \ \ \forall\, x \in K$.

We say that f is weakly inward in case $f(x) \in \overline{I_K(x)} \ \ \forall\, x \in K$.

The proof of the fact that the Leray-Schauder condtion is weaker than the inwardness condition defined above, can be found for instance in [2] in the case K is convex.

The following counterexample shows that the converse is not true :

COUNTEREXAMPLE 1. Let $E = R^2$ with the euclidean norm

$$K := \{(a, b) \in E : a^2 + b^2 \leq 1\},\ f(a, b) := (a, b + |a|).$$

We show that f satisfies (1) with $w = 0$, i.e., $f(x) = mx$ implies $m \leq 1$.

Let $x = (a, b) \in \partial K$. Then $x = (a, \pm (1 - a^2)^{1/2})$ and $f(x) = (a, \pm (1 - a^2)^{1/2} + |a|)$.

Moreover let $f(x) = mx$, that is

$$\begin{cases} a = ma & (2) \\ \pm (1 - a^2)^{1/2} + |a| = m(\pm (1 - a^2)^{1/2}) & (3) \end{cases}$$

Now, if $a \neq 0$, furnish $m = 1$ and then (3) yields $a = 0$, a contradiction ! So $f(x) = mx$ only if $x = (0, \pm 1)$ and in this case (3) implies $m = 1$.

On the other had, one can see immediately that f is not weakly inward.

Some extensions of the (Weak) inwardness and Leray-Schauder condtions are useful also in the search for fixed points for multivalued mappings

(Results and references can be found for example in Deimling [2], [3]).

For example, if $T : K \to 2^E$ is a multivalued mapping, a suitable reformulation of (1) is

$\exists\, w \in K^0$ such that $\forall\, x \in \partial K,\ w + m(x - w) \in Tx$ implies $m \le 1$ ... (4)
and a suitable reformulation of inwardness is

$$T\,x \cap I_K(x) \neq \phi \quad \forall\, x \in K. \qquad \text{... (5)}$$

However, while in the univoque case (5) implies (4) whenever K is convex, but in the multivalued case this is no longer true :

COUNTEREXAMPLE 2.

Take E, K as in counterexample 1.

$$T(a, b) := (a, [b, b + 1]).$$

Then $(a, b) \in T(a, b)$ for $(a, b) \in K$, so T satisfies (5). We show that T does not satisfy (4).

Let $w \in K^0$, $w = (a_0, b_0)$. Let $x = x(w) = (a_0, (1 - a_0^2)^{1/2}) \in \partial k$, $z = (a_0, (1 - a_0^2)^{1/2} + 1)$, $m = 1 + ((1 - a_0^2)^{1/2} - b_0)^{-1} > 1$. Then $w + m(x - w) \in Tx$ and this means (4) is not satisfied.

REMARK. We observe that the map of Counterexample 2 verifies a wak Leray-Schauder condtion : $\exists\, w \in K^0$ such that $\forall\, x \in \partial K$, $w + m(x - w) \in Tx$ implies $\exists\, c \le 1$ such that $w + c(x - w) \in Tx$. The last condtion was introduced by De Pascale-Guzzardi in [4] and in our opinion it is the true formulation of the Leray-Schauder condtion in the context of multivalued maps with convex values.

A similar situation occurs for nonexpansive and pseudocontractive maps : it is immediate to see that, in the univoque case, nonexpansivity implies pseudocountractivity, i.e. $\|f(x) - f(y)\| \le \|x - y\|$ implies

$$\|x - y\| \le \|(1 + r)\,(x - y) - r(f(x) - f(y))\| \quad \forall\, x, y \in K,\ r > 0.$$

In the multivalued case it is no longer true :

COUNTEREXAMPLE 3.

Let $E = R$, $K = E$, $Tx := [x, x + 1]$. T is nonexpansive (i.e. $D(Tx, Ty) \le \|x - y\|$ but not pseudocontractive (i.e.

$$\|x - y\| \le \|(1 + r)\,(x - y) - r(u - v)\|$$

$\forall\, x, y \in E,\ u \in Tx,\ v \in Ty, r > 0$ is not satisfied).

COUNTEREXAMPLES 4 AND 5

In the last years, the interest in optimization theory for the multivalued maps T satisfying Fix $(T) = \{z\}$ and $\{z\} = Tz$, has prompted a corresponding interest in Fixed Point Theory, since in [5] it has been shown that the maximization of a multivalued map T with respect to a cone, which subsumes ordinary and Pareto optimization, is equivalent to a fixed point problem of determining z such that $Tz = \{z\}$.

With this purpose, a sufficient condition that ensures $Fix(T) = \{z\}$ and $Tz = \{z\}$ for a multivalued map $T : X \to CB(X)$ is, for example

$$\delta(Tx, Ty) \le q \max\{d(x, y), \delta(x, Tx), \delta(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}$$
with $0 < q < 1$ ([6]).

On the other hand, if the multivalued mapping T satisfies

$$\delta(Tx, Ty) \le Ad(x, Tx) + Bd(y, Ty) + cd(x, y) \qquad \text{... (6)}$$

with $A, B \in]\,0, \infty[$ and $c \in]\,0, 1[$

or

$$\delta(Tx, Ty) \le a\delta(x, Tx) + b\delta(y, Ty) + cd(x, y) \qquad \text{... (7)}$$

then $Fix(T) \ne \phi$ obviously implies $Fix(T) = \{z\}$ and $Tz = \{z\}$.

A sufficient condition which ensures $Fix(T) \ne \phi$ is, for example

$$D(Tx, Ty) \le q \max\{d(x, y), \tfrac{1}{2}[d(x, Tx) + d(y, Ty)], \tfrac{1}{2}[d(x, Ty) + d(y, Tx)]\}$$

with $0 < q < 1$ [7]. ... (8)

However we note that the map $T: N \to N$ defined by $T0 = T1 = 0$, $Tn := \{0, 1\}, n \ge 2$, satisfies both (6) and (7) with $A = 10$, $B = 10, a = 0.4, b = 0.5, c = 0.9$ but it does not satisfy (8), although Fix $(T) \ne 0$. one could suspect that the condition (6) or (7) always implies $Fix\ (T) \ne \phi$. This is not true, as the following example shows :

COUNTEREXAMPLE 4.

Let $X = N, T0 := 1, Tn := \{0, 1\}$ for $n \ge 2$. Then (6) and (7) both are satisfied with $A = 20, B = 20, a = 0.4, b = 0.5, c = 0.9$. Nevertheless Fix $(T) = \phi$.

At this point, we recall that interesting results are known for the weak contraction single valued mappings (i.e. $d(f(x), f(y)) < d(x, y)$). Such contractions were studied first by Edelstein in [8].

In the case of a multivalued map T, taken into account that (6) and (7) are not sufficient to guarantee $Fix\ (T) \ne \phi$, not even if the domain of T is a compact set, one can suspect that the following condition is sufficiently strong :

$$\delta(Tx, Ty) < d(x, y) \text{ for } x \ne y. \qquad \text{... (9)}$$

Actually, such a condition is too strong, in the sense that nearly always do not exist "authentic" multivalued maps that satisfy (9) as shown in the following lemma :

LEMMA Let X be a metric space without isolated points. Let $T: X \to 2^X$ be a multivalued mapping satisfying (9). Then $T = f$ is a single valued map.

PROOF. Let $x \in X$ and $y_n \in B(x, 1/n)$, $y_n \ne x$. Then (9) implies

$$\delta(Ty_n, Tx) \to 0 \text{ for } n \to \infty \qquad \text{... (10)}$$

and from this its follows that Tx is a singleton. Indeed, if there exist $z, w \in Tx$ with $z \ne w$, we put $r = \dfrac{1}{2} d(z, w)$ and we show that

$$\delta(Ty_n, Tx) \ge r \text{ for each } n \qquad \text{... (11)}$$

contradicting (10). If fact, if $\forall\ v \in Ty_n$ it results $d(z, v) \geq r$, then (11) is obviously true. On the contrary, if there exists $v \in Ty_n$ such that $d(z, v) < r$ then $2r = d(z, w) \leq d(z, v) + d(v, w) < r + d(v, w)$, i.e. $d(v, w) > r$ and this still yields (11) since $(Ty_n, Tx) \geq d(v, w)$.

On the contrary, if the metric space X has isolated points, there exist multivalued mappings T satisfying (9) and in such a case $Fix(T)$ is a singleton ($Fix(T) = \{z\}$) of course. But that which is not verified in general is $Tz = \{z\}$ as the following counterexample shows :

COUNTEREXAMPLE 5.

$$X = \{0, 1, 2/3\} \cup \{2/3 - \sum_{k=1}^{n} 10^{-k},\ n \geq 1\}.\ T : X \to 2^X,\ T0 := \{0, 1\},$$

$$T1 := 2/3) := 2/3 - 10^{-1},\ T(2/3 - \sum_{k=1}^{n} 10^{-k}) = 2/3 - \sum_{k=1}^{n+1} 10^{-k}.$$

Then T satisfies (7), $Fix(T) = \{0\}$ but $T0 \neq \{0\}$.

COUNTEREXAMPLE 6

Finally, let H be a Hilbert space and K a closed convex subset of H. For a map $T : K \to KC(H)$ we define $\forall\ x \in K$

$$\hat{T}x := \{y \in Tx : d(y, K) = d(Tx, K)\},$$

the Fan's best approximation from Tx to K.

One can see that the convexity of Tx implies the convexity of $\hat{T}x$ and $P_K\hat{T}x$, is projection on K[9].

But without the assumption of convexity of Tx, the reader is invited to find examples of mappings such that T is nonexpansive while $\hat{T}$ is not.

In [8] was an open question if the nonexpansivity of $T : K \to KC(H)$ implies the nonexpansivity of $\hat{T}$. The following counterexample, due to *H.K.Xu*, negatively answers to such question :

COUNTEREXAMPLE 6.

$H = R^2$ with the euclidean norm. $K =$ triangle with vertices $D(\frac{1}{2}, 0), A(1, 0)$ and $C(\frac{1}{2}, \frac{1}{2})$. Define $T : K \to KC(H)$ as follows. Let $z = (x, y)$ be any point in K and let z' be the symmetric point of z with respect to the segment CD. Let P be the projection of z' onto the x-axis. Then we define

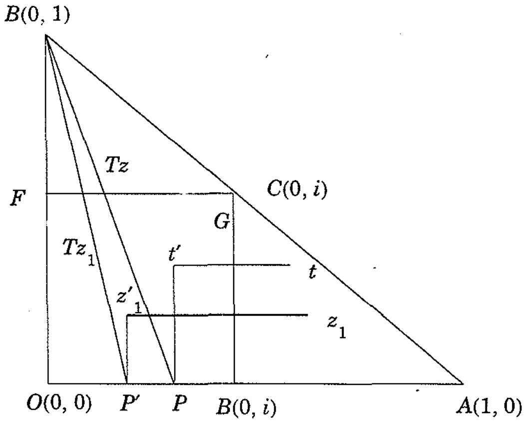
$Tz :=$ the segment BP linking B and P.

Then if $z = (x, y)$ and $z_1 = (x_1, y_1)$, we have $D(Tz, Tz_1) = |P - P'| \leq \|z - z_1\|$,

i.e. T is nonexpansive. It is also easily seen that

$$Tz = \begin{cases} P & \text{if } z \neq A \\ OF & \text{if } z = A. \end{cases}$$

It follows that for G on the open segment CD,



$$D(\hat{T}G, \hat{T}A) = \sup_{z \in 0F} d(z, D) = d(F, D) = d(C, A), d(G, A),$$ so T is not nonexpansive.

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