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INTEGRALS INVOLVING A GENERAL CLASS OF MULTIVARIABLE POLYNOMIALS, JACOBI POLYNOMIALS, AND FOX'S *H*-FUNCTION

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ABSTRACT

Here we evaluate four integrals involving various products of general class of multivariable polynomials introduced earlier by H.M. Srivastava and M. Garg, Jacobi ploynomials, and Fox's *H*- function. Our results are quite general in character and provide interesting unifications and generalizations of a large number of (known or new) integral formulae.

1. Introduction

Recently Kalla, Conde and Luke [2], Kalla [3] and Kant and Koul [4] have established certain integrals involving Jacobi polynomials, generalized Jacobi functions, a general class of polynomials and Fox's *H*-function. In an attempt to unify and extend these results we have evaluated four finite integrals involving the product of Jacobi polynomials, a general class of multivariable polynomials and Fox's *H*-function. The technique followed is essentially that of Kalla et al.([2],[3]).

The general class of polynomials $S_n^m[x]$ introduced by Srivastava [7],p.1, Eq. (1)] has further been generalized to a multivariable polynomial in the following manner (Srivastava and Garg [8] :

$$S_n^{m_1, \dots, m_r}(x_1, \dots, x_r) = \sum_{k_1, \dots, k_r = 0}^{m_1 k_1 + \dots + m_r k_r \le n} (-n)_{m_1 k_1 + \dots + m_r k_r} A(n\,; k_1, \dots, k_r) \frac{x_1^{k_1} \dots x_r^{k_r}}{k_1! \dots k_r!} \qquad \dots (1.1)$$

where $m_1, \dots, m_r$ are arbitrary positive integers and the coefficients $A(n\,; k_1, \dots, k_r)$ $(n, k_i \ge 0, i = 1, \dots, r)$ are arbitrary constants real or complex.

To facilitate the derivation of our main integrals given in the next section we shall require an integral contained in the following

Lemma If $\mu, \nu, \gamma_i, \delta_i$ $(i = 1, ..., r)$ are all non-negative real numbers (not all zero simultaneously), Re $(\alpha) > -1$, Re$(\beta) > -1$,

$$\text{Re}(\rho) + \mu \min_{l \le j \le M} [\text{Re}(b_j/B_j)] > -1, \text{Re}(\sigma) + \nu \min_{l \le j \le M} [\text{Re}(b_j/B_j)] > -1,$$

$A > 0, \delta > 0, \; |arg(z)| < \frac{1}{2} A\pi$, then

$$\int_0^t (t-x)^\rho x^\sigma p_u^{(\alpha,\beta)}(1 - z' x) S_n^{m_1, \dots, m_r}(z_1(t-x)^{\gamma_1} x^{\delta_1}, \dots, z_r(t-x)^{\gamma_r} x^{\delta_r})$$

$$H_{P,Q}^{M,N}\left[z(t-x)^\mu x^\nu \left| \begin{matrix} (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1\,Q} \end{matrix} \right.\right] dx$$

$$= t^{\rho+\sigma+1} \sum_{k_1, \dots, k_r = 0}^{m_1 k_1 + \dots + m_r k_r \le n} \sum_{w=0}^{u} \frac{(-n)_{m_1 k_1 + \dots + m_r k_r} A(n; k_1, \dots, k_r)}{k_1! \dots k_r!}$$

$$\frac{\Gamma(\alpha + u + 1)(-u)_w (\alpha + \beta + u + 1)_w}{\Gamma(\alpha + w + 1)\, u!\, w!} (\tfrac{1}{2} z' t)^w \prod_{i=1}^r (z_i t^{\gamma_i + \delta_i})^{k_i}$$

$$H_{p+2,Q+1}^{M,N+2}\left[z t^{\mu+\nu} \left| \begin{matrix} (-\rho - \sum_{i=1}^r \gamma_i k_i, \mu), (-\sigma - w - \sum_{i=1}^r \delta_i k_i, \nu), (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,Q}, (1 - \rho - \sigma - w - \sum_{i=1}^r (\gamma_i + \delta_i) k_i, \mu + \nu) \end{matrix} \right.\right] \quad \dots (1.2)$$

$$= t^{\rho+\sigma+1} \sum_{k_1, \dots, k_r = 0}^{m_1 k_1 + \dots + m_r k_r \le n} \sum_{w=0}^{u} \sum_{h=0}^{M} \sum_{R-0}^{\infty} \frac{(-n)_{m_1 k_1 + \dots + m_r k_r}}{k_1 \dots k_r!} A(n; k_1, \dots, k_r)$$

$$\frac{\prod_{i=1}^r (z_i t^{\gamma_i + \delta_i})^{k_i} \Gamma(\alpha + u + 1)(-u)_w (\alpha + \beta + u + 1)_w \Gamma(1 + \rho + \sum_{i=1}^r \gamma_i k_i + \mu \xi_{h,R})}{\Gamma(\alpha + w + 1)\, u!\, w!\, R!\, \Gamma(2 + \rho + \sigma + w + \sum_{i=1}^r (\gamma_i + \delta_i) k_i + (\mu + \nu) \xi_{h,R})}$$

$$\frac{\Gamma(1 + \sigma + w + \sum_{i=1}^r \delta_i k_i + \nu \xi_{h,} R)}{\Gamma(1 + \sigma + w + \sum_{i=1}^r \delta_i k_i + \nu \xi_{h,} R)} f(\xi_{h,R}) (\tfrac{1}{2} z', t)^w (z\, t^{\mu+\nu})^{\xi_{h,R}} \quad \dots (1.3)$$

$$= \sum_{k_1, \dots, k_r = 0}^{m_1 k_1 + \dots + m_r k_r \le n} \sum_{w=0}^{u} \sum_{h=0}^{M} \sum_{R=0}^{\infty} g(w, h, R) \dots \text{(say)} \quad \dots (1.4)$$

$$\text{where } \xi_{h,R} = \frac{b_h + R}{B_h}, \quad (R = 0, 1, 2, \ldots) \qquad \ldots (1.5)$$

$$f(\xi_{h,R}) = \frac{\prod_{\substack{j=1 \\ j \neq h}}^{M} \Gamma(b_j - B_j \xi_{h,R}) \prod_{j=1}^{N} \Gamma(1 - a_j - A_J \xi_{h,R})}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j + B_j \xi_{h,R}) \prod_{j=N+1}^{P} \Gamma(a_j - A_j \xi_{h,R}) B_h} \qquad \ldots (1.6)$$

$$A = \sum_{j=1}^{N} A_j - \sum_{j=N+1}^{P} A_j + \sum_{j=1}^{M} B_j - \sum_{j=M+1}^{Q} B_j \qquad \ldots (1.7)$$

$$\text{and } \delta = \sum_{j=1}^{Q} B_j - \sum_{j=1}^{P} A_j . \qquad \ldots (1.8)$$

The integral (1.2) is a special case of an integral recently established by the author ([6], p.10, Eq. (2.1). The value of the integral in the form given by (1.3) can be obtained easily by using the following series expansion for Fox's H-function due to Braaksma [1], see also Srivastava et al. ([9], p.12, Eq. (2..2.4))

$$H_{P,Q}^{M,N}[x] = H_{P,Q}^{M,N}\left[x \,\middle|\, \begin{matrix} (a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q} \end{matrix} \right] = \sum_{h=1}^{M} \sum_{R=0}^{\infty} f(\xi_{h,R}) x^{\xi_{h,R}} \qquad \ldots (1.9)$$

where $\xi_{h,R}$ and $f(\xi_{h,R})$ are defined by (1.5) and (1.6) respectively. $H_{P,Q}^{M,N}[x]$ stands for the well-known Fox's H-function. For details of this function one can refer to the book by Srivastava et al. [9].

2. Main Integrals

Let $\mu, \nu, \gamma_i, \delta_i$ are all non-negative real numbers (not all zero simultaneously), $\delta > 0, A > 0$, $|\arg(z)| < \frac{1}{2}A\pi, A$ and δ being given by (1.7) and (1.8) , respectively.

$$\mathrm{Re}(\rho) + \mu \min_{1 \le j \le M} \{\mathrm{Re}(b_j/B_j)\} > -1$$

$$\mathrm{Re}(\sigma) + \nu \min_{l \le j \le M} \{\mathrm{Re}(b_j/B_j)\} > -1,$$

and

$$F(x) = (t-x)^{\rho} x^{\sigma} P_u^{(\alpha,\beta)}(1 - z'x)\, S_n^{m_1, \ldots, m_r}(z_1 (t-x)^{\gamma_1} x^{\delta_1}, \ldots, z_r (t-x)^{\gamma_r} x^{\delta_r})\, H_{P,Q}^{M,N}\left[z(t-x)^{\mu} x^{\nu} \,\middle|\, \begin{matrix} (a_j, A_j)_{1,P} \\ b_j, B_j)_{1,Q} \end{matrix} \right] \qquad \ldots (2.1)$$

then

(i) $\int_0^t F(x) \log (t-x)\, dx = \sum_{k_1, \ldots, k_r = 0}^{m_1 k_1 + \ldots + m_r k_r \le n} \sum_{w=0}^{u} \sum_{h=1}^{M} \sum_{R=0}^{\infty} g(w, h, R)$

$$\left[\log t + \psi\left(1 + \rho + \sum_{i=1}^{r} \gamma_i k_i + \mu \xi_{h,r}\right) \psi\left(2 + \rho + \sigma + w + \sum_{i=1}^{r} (\gamma_i + \delta_i) + k_i + (\mu + \nu)\xi_{h,R}\right)\right] \quad \ldots (2.2)$$

(ii) $\int_0^t F(x) \log x\, dx = \sum_{k_1, \ldots, k_r = 0}^{m_1 k_1 + \ldots + m_r k_r \le n} \sum_{w=0}^{u} \sum_{h=1}^{M} \sum_{R=0}^{\infty} g(w, h, R).$

$$\left[\log t + \psi\left(1 + \sigma + \sum_{i=1}^{r} \delta_i k_i + \nu \xi_{h,R} - \psi\left(2 + \rho + 2 + w + \sum_{i=1}^{r} (\gamma_i + \delta_i) k_i + (\mu + \nu)\xi_{h,R}\right)\right)\right] \quad \ldots (2.3)$$

(iii) $\int_0^t F(x) \log [(t-x)x]\, dx = \sum_{k_1, \ldots, k_r = 0}^{m_1 k_1 + \ldots + m_r k_r \le n} \sum_{w=0}^{u} \sum_{h=1}^{M} \sum_{R=0}^{\infty} g(w, h, R)$

$$\left[\log t^2 + \psi\left(1 + \rho + \sum_{i=1}^{r} \gamma_i k_i + \mu \xi_{h,R}\right) + \psi\left(1 + \sigma + w + \sum_{i=1}^{r} \delta_i k_i + \nu \xi_{h,r}\right) - 2\psi\left(2 + \rho + \sigma + w + \sum_{i=1}^{r} (\gamma_i + \delta_i) k_i + (\mu + \nu)\xi_{h,R}\right)\right] \quad \ldots (2.4)$$

(iv) $\int_0^t F(x) \log \left[\frac{t-x}{x}\right] dx = \sum_{k_1, \ldots, k_r = 0}^{m_1 k_1 + \ldots + m_r k_r \le n} \sum_{w=0}^{u} \sum_{h=1}^{M} \sum_{R=0}^{\infty} g(w, h, R)$

$$\left[\psi\left(1 + \rho + \sum_{i=1}^{r} \gamma_i k_i + \mu \xi_{h,R}\right) - \psi\left(1 + \sigma + w + \sum_{i=1}^{r} \delta_i k_i + \nu \xi_{h,R}\right)\right] \quad \ldots (2.5)$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ and $g(w, h, R)$ and $\xi_{h,R}$ are given by (1.4) and (1.5) respectively. Also the series occurring on the right-hand side of integrals (2.2) through (2.5) converge absolutely.

Proof : The results in (2.2) and (2.3) can be obtained by taking the partial derivatives of both sides of (1.3) with respect to ρ and σ. respectively. The integral (2.4) is obtained by adding (2.2) and (2.3) and the integral (2.5) is obtained by subtracting (2.3) from (2.2).

3. Particular Cases

At the outset, we should remark that our integral formulae are quite general in character. Indeed, these results can suitably be

specialized to a number of known and new integrals involving a large spectrum of various classes of polynomials and elementary special functions. However, we mention here only a few special cases of (2.2).
(i) If we put

$$A(n;k_1,\ldots,k_r)=\frac{\prod_{j=1}^{E}(e_j)_{k_1\theta_j'+\ldots+k_r\theta_j^{(r)}}\prod_{j=1}^{U'}(u_j')_{k_1\phi_j'}\cdots\prod_{j=1}^{U^{(r)}}(u_j^{(r)})_{k_r\varphi_j^{(r)}}}{\prod_{j=1}^{D}(d_j)_{k_1\tau_j'+\ldots+k_r\tau_j^{(r)}}\prod_{j=1}^{V'}(v_j')_{k_1\zeta_j'}\cdots\prod_{j=1}^{V^{(r)}}(v_j^{(r)})_{k_r\zeta_j^{(r)}}} \quad \ldots (3.1)$$

in (2.1) we arrive easily at the following interesting integral formula

$$\int_0^t (t-x)^{\rho}x^{\sigma}P_u^{(\alpha,\beta)}(1-z'x)\log(t-x)\,F^{1+E:U',\ldots,U^{(r)}}_{D:V',\ldots,V^{(r)}}\left[\begin{matrix}[-n:m_1,\ldots,m_r],\\ [(e):\theta',\ldots,\theta^{(r)}]:[(U'):\varphi'];\ldots;[(U^{(r)}):\varphi^{(r)}]\\ [(d):\tau',\ldots,\tau^{(r)}]:[(V'):\zeta']:\ldots:[(V^{(r)}):\zeta^{(r)}]\end{matrix}\; z_1(t-x)^{\gamma_1}x^{\delta_1},\ldots, z_r(t-x)^{\gamma_r}x^{\delta_r}\right]H^{MN}_{PQ}\left[z(t-x)^{\mu}x\left|\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right]dx$$

$$=\sum_{k_1,\ldots,k_r=0}^{m_1k_1+\ldots+m_rk_r\le n}\;\sum_{w=0}^{u}\;\sum_{h=1}^{M}\;\sum_{R=0}^{\infty} n(w,h,R)\Big[\log t+\psi\Big(1+\rho+\sum_{i=1}^{r}\gamma_ik_i+\mu\xi_{h,R}\Big)-\psi\Big(2+\rho+\sigma+w+\sum_{i=1}^{r}(\gamma_i+\delta_i)k_i+(\mu+\nu)\xi_{h,R}\Big)\Big] \quad \ldots (3.2)$$

where $\eta(w,h,R)$ is obtained by putting the value of $A(n;k_1,\ldots,k_r)$ from (3.1) in $g(w,h,R)$.

(ii) Again if we take $m_i=0, z_i\to 0\ (i=2,\ldots,r)$ (2.2) and replace $A(n;k_1,0,\ldots,0)$ by $A_{n,k}$ therein, we easily arrive at the following integral

$$\int_0^t (t-x)^{\rho}x^{\sigma}P_u^{(\alpha,\beta)}(1-z'x)\,S_n^m[z_1(t-x)^{\gamma}x^{\delta}]\log(t-x)\,H^{(M,N)}_{PQ}\left[z(t-x)^{\mu}x^{\nu}\left|\begin{matrix}(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right]dx$$

$$=\sum_{k=0}^{[n/m]}\sum_{w=0}^{u}\sum_{h=0}^{M}\sum_{R=0}^{\infty}t^{\rho+\sigma+1}\frac{(-n)_{mk}A_{n,k}\Gamma(\alpha+u+1)(-u)_w(\alpha+\beta+u+1)_w}{k!\,\Gamma(\alpha+w+1)\,u!\,w!\,R!}$$

$$\frac{\Gamma(1+\rho+\gamma k+\mu\xi_{h,R})\,\Gamma(1+\sigma+w+\delta k+\nu\xi_{h,R})}{(2+\rho+\sigma+w+(\gamma+\delta)k+(\mu+\nu)\,\xi(h,R)}\Big[\log t+\psi(1+\rho+\gamma k+\mu\xi_{h,R})$$

$$- \psi(2+\rho+\sigma+w+(\gamma+\delta)k+(\mu+\nu)\,\xi_{h,R})\Big] f(\xi_{h,R})(z_1 t^{\gamma+\delta})^k \left(\tfrac{1}{2}z' t\right)^w (z t^{\mu+\nu})^{\xi_{h,R}} \qquad \ldots (3.3)$$

where

$$S_n^m(x) = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k\,!} A_{n,k} x^k \qquad \ldots (3.4)$$

is a general class of polynomials introduced by Srivastava ([7], p.1, Eq. (1). This general class of polynomials include almost all the wellknown orthogonal polynomial and generalized hypergeometric polynomials available in the literature. For example

(iii) If we put $z_1 = 1, A_{n,k} = \dfrac{\Gamma(1+\eta+\lambda n)}{n\,!\,\Gamma(1-\eta+\lambda k)}$ ... (3.5)

in (3.3) , replace γ and δ by γ/λ and δ/λ respectively, we get the following interesting integral :

$$\int_0^t (t-x)^\rho x^\sigma P_u^{(\alpha,\beta)}(1-z'x)\log(t-x)\, Z_n^m[(t-x)^\gamma x^\delta;\lambda]$$

$$H_{P\,Q}^{M\,N}\left[z(t-x)^\mu x^\nu \left|\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right]dx = \sum_{k=0}^{[n/m]}\sum_{w=0}^{u}\sum_{h=0}^{M}\sum_{R=0}^{\infty}$$

$$t^{\rho+\sigma+1}\frac{(-n)_{mk}\,\Gamma(1+\eta+\lambda n)\,\Gamma(\alpha+u+1)\,(-u)_w\,(\alpha+\beta+u+1)_w}{k\,!\,n\,!\,\Gamma(1-\eta+\lambda k)\,\Gamma(\alpha+w+1)\,u\,!\,w\,!\,R\,!}$$

$$\frac{\Gamma(1+\rho+\frac{\gamma}{\lambda}k+\mu\,\xi_{h,R})\,\Gamma(1+\sigma+w+\frac{\delta}{\lambda}k+\nu\,\xi_{h,R})}{\Gamma(2+\rho+\sigma+w+(\frac{\gamma+\delta}{\lambda})k+(\mu+\nu)\,\xi_{h,R})}\Bigg[\log t +$$

$$\psi(1+\rho+\frac{\gamma}{\lambda}k+\mu\,\xi_{h,R}) - \psi\left(2+\rho+\sigma+w+\left(\frac{\gamma+\delta}{\lambda}\right)k+(\mu+\nu)\,\xi_{h,R}\right)\Bigg]$$

$$f(\xi_{h,R})\, t^{(\gamma+\delta)k}\left(\tfrac{1}{2}z't\right)^w (zt^{\mu+\nu})^{\xi_{h,R}} \qquad \ldots (3.6)$$

where $Z_n^m(x;\lambda)$ stands for the Kohnauser biorthogonal polynomial (Konhauser [5]).

(iv) If we take $u = 0$, $t = 1$ in (3.3) and replacing x by $\dfrac{1+y}{2}$ therein we easily get the integral recently given by Kant and Kaul ([4], p. 6, Eq. (15). The other integrals (16),(17) and (18) due to them are also special cases of our main integrals.

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