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COMMON FIXED POINTS OF TWO PAIRS OF SEQUENCES OF MAPPINGS

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ABSTRACT

A common fixed point theorem for two pairs of sequences of pairwise weakly commuting selfmappings of a complete metric space satisfying Meir and Keeler type contractive condition is obtained. The existence of the fixed point has been established under the assumption of continuity of only one of the mappings. Our work generalizes several fixed point theorems concerning contractive mappings.

1. INTRODUCTION

The study of common fixed points of mappings satisfying some contractive type condition has been at the centre of vigorous research activity and a number of interesting results have been obtained by various authors. Most of these results either deal with commuting mappings or assume the notion of weak commutativity of mappings introduced by Sessa [8]. Jungck [2] introduced the notion of compatibility of mappings, also called asymptotic commutativity by Tivari and Singh [9] in an independent formulation. It was claimed that weak commutativity implies compatibility [3], [9] but not conversely [7]. However, in a review of [3] (Mathematical Review 89 h : 54030) Singh has produced an example to show the existence of a weakly commuting pairs of mappings satisfying a contractive condition for which there exists no sequence of points satisfying the condition of compatibility.

In this paper we obtain a common fixed point theorem for two pairs of sequences of pairwise weekly commuting mappings satisfying a Meir and Keeler type contractive condition. We have assumed the continuity of only one of the mappings. The mapping condition studied by us is a generalization of mapping condition 10 of Rhoades [6]. Our work generalizes the results due to Fisher [1], Pant [4], Park and Bae [5] and some other results.

2. RESULTS

If (X, d) be a metric space, two selfmappings F and G of X are called weakly commuting provided $d(FGX, GFx) \le d(Fx, Gx)$ for each x in X.

Theorem. Let $\{P_i\}, \{S_i\}, \{Q_j\}$ and $\{T_j\}$, $i, j = 1, 2, 3, \ldots,$ be sequences of mapings of a complete metric space (X, d) into itself satisfying the conditions :

Given $\varepsilon > 0$, there exists an $h(\varepsilon) > 0$, $h(\varepsilon)$ being nondecreasing function of ε, such that for all x, y in X

$$\varepsilon \leq \max \{d(S_ix, T_jy), d(P_ix, S_ix), d(Q_jy, T_jy)\} < \varepsilon + h$$

$$\Rightarrow \quad d(P_ix, Q_jy) < \varepsilon \qquad \ldots (1)$$

$$P_ix = Q_jy \quad \text{whenever} \quad P_i x = S_ix,\ Q_jy = T_jy. \qquad \ldots (2)$$

Let P_i and S_i commute weakly and Q_j and T_j commute weakly for every value of i and j respectively. If the ranges of T_j and S_i respectively contain the ranges of P_i and Q_j for every value if i and j and if one of the mappings is continuous then the sequences $\{P_i\}, \{S_i\}, \{Q_j\}$ and $\{T_j\}$ have a unique common fixed point which is also the unique common fixed point of $\{P_i\}$ and $\{S_i\}$ and of the sequences $\{Q_j\}$ and $\{T_j\}$.

Proof. First, with the help of (1), we note that for all x, y in X such that $P_ix \neq S_ix$, $Q_jy \neq T_jy$, $i, j = 1, 2, 3, \ldots,$

$$d(P_ix, Q_jy) < \max \{d(S_ix, T_jy), d(P_ix, S_ix), d(Q_jy, T_jy)\}. \qquad \ldots (3)$$

Secondly, the nondecreasing character of $h(\varepsilon)$ implies that given $\varepsilon > 0$, there exists $\varepsilon_0 > 0$, such that $\varepsilon_0 < \varepsilon < \varepsilon_0 + h(\varepsilon_0)$ or equivalently.

$$\max\{d(S_ix, T_jy), d(P_ix, S_ix), d(Q_jy, T_jy)\} = \varepsilon$$

$$\Rightarrow \quad d(P_ix, Q_jy) < \varepsilon_0, \varepsilon_0 0 < \varepsilon. \qquad \ldots (4)$$

Let us arbitrarily select a pair of integers i and j and let x_0 be any point in X, Choose a sequence of points $\{x_n : n = 0, 1, 2, \ldots\}$ in X defined by $P_ix_{2n} = T_jx_{2n+1}$ and $Q_jx_{2n+1} = S_ix_{2n+2}$. This can be done since the ranges of T_j and S_i respectively contain the ranges of P_i and Q_j. We can assume that $P_i x_{2n} \neq Q_j x_{2n+1}$ and $Q_jx_{2n+1} \neq P_ix_{2n+2}$ for every value of n, otherwise the existence of the common fixed point is easy to establish. Then from (3), we obtain

$$d(P_ix_{2n}, Q_jx_{2n+1}) < d(Q_jx_{2n-1}, P_ix_{2n}), \qquad \ldots (5)$$

and $$d(Q_jx_{2n-1}, P_ix_{2n}) < d(P_ix_{2n-2}, Q_jx_{2n-1}). \qquad \ldots (6)$$

Also for any integer $p > 0$

$$d(Q_jx_{2n+1}, P_ix_{2(n+p)+2}) < \max \{d(P_ix_{2n}, Q_j x_{2(n+p)+1}),$$
$$d(P_ix_{2n}, Q_jx_{2n+1})\} \qquad \ldots (7)$$

and $d(P_i x_{2n}, Q_j x_{2(n+p)+1})$

$$< \max \{d(Q_j x_{2n-1}, P_i x_{2n}), d(Q_j x_{2n-1}, P_i x_{2(n+p)})\}. \quad \dots (8)$$

In view of (5) and (6) we claim that $\lim_{n \to \infty} d(P_i x_{2n}, Q_j x_{2n+1}) = 0 = \lim_{n \to \infty} d(Q_j x_{2n+1}, P_i x_{2n+2})$. For, if not, suppose for instance $\lim_{n \to \infty} d(P_i x_{2n}, Q_j x_{2n+1}) = r, r > 0$. Then given $h > 0$ there exists a positive integer N such that for each integer $m \geq N$, we have

$$r \leq d(P_i x_{2m}, Q_j x_{2m+1}) < r + h \quad \dots (9)$$

or

$$r \leq \max \{d(S_i x_{2m+2}, T_j x_{2m+1}), d(P_i x_{2m+2}, S_i x_{2m+2}), d(Q_j x_{2m+1}, T_j x_{2m+1})\} < r + h. \quad \dots (10)$$

Selecting h in (10) in accordance with (1), for each $m \geq N$ we obtain $d(P_i x_{2m+2}, Q_j x_{2m+1}) < r$ and so $d(P_i x_{2m+2}, Q_j x_{2m+3}) < r$, which contradicts (9). Therefore

$$\lim_{n \to \infty} d(P_i x_{2n}, Q_j x_{2n+1}) = 0 = \lim_{n \to \infty} d(Q_j x_{2n+1}, P_i x_{2n+2}) \quad \dots (11)$$

Also, in view of (7) (8) and (11), it follows that

$$\lim_{n \to \infty} d(P_i x_{2n}, Q_j x_{2(n+p)+1}) = \lim_{n \to \infty} d(Q_j x_{2n+1}, P_i x_{2(n+p)+2}).$$

Following similar argument, as used to evaluate limit (11),it follows easily that

$$\lim_{n \to \infty} d(P_i x_{2n}, Q_j x_{2(n+p)+1}) = 0 = \lim_{n \to \infty} d(Q_j x_{2n+1}, P_i x_{2(n+p)+2}).$$

Hence $\{P_i x_0, Q_j x_1, \dots, P_i x_{2n}, Q_j x_{2n+1}, \dots\}$ is a Cauchy sequence in the complete metric space X and so has a limit z in X.

Also, the sequence $\{P_i x_{2n} = T_j x_{2n+1}\}$ and $\{Q_j x_{2n+1} = S_i x_{2n+2}\}$ converge to z.

Let us now suppose that the mapping S_i is continuous. Then, since P_i and S_i commute weakly, the sequences $\{P_i S_i x_{2n}\}$ and $\{S_i S_i x_{2n}\}$ coverage to $S_i z$. We claim that $z = S_i z$. For if $z \neq S_i z$ then the inequality

$$d(P_i S_i x_{2n}, Q_j x_{2n+1}) < \max \{d(S_i S_i x_{2n}, T_j x_{2n+1}), d(P_i S_i x_{2n}, S_i S_i x_{2n}), d(Q_j x_{2n+1}, T_j x_{2n+1})\}$$

on letting $n \to \infty$ and in view of (4) leads to $d(z, S_i z) < d(z, S_i z)$, a contradiction. Therefore $z = S_i z$. Similarly the inequality

$$d(P_i z, Q_j x_{2n+1}) < \max\{d(S_i z, T_j x_{2n+1}), d(P_i z, S_i z), d(Q_j z_{2n+1}, T_j x_{2n+1})\}$$

yields $z = P_i z$. This means that there exists a point z_0 in X such that $S_i z = z = P_i z = T_j z_0$, as the range of P_i is contained in the range of T_j. Moreover, the inequality

$$d(P_i z, Q_j z_0) < \max\{d(S_i z, T_j z_0), d(P_i z, S_i z), d(Q_j z_0, T_j z_0)\}$$

yields $z = Q_j z_0 = T_j z_0$. This equation, in view of weak commutativity of Q_j and T_j, implies $Q_j z = T_j z$. Finally, the relation $Q_j z = T_j z$ together with (3) leads to $z = Q_j z = T_j z$.

We have therefore proved that z is a common fixed point of P_i, Q_j, S_i and T_j for the arbitrarily chosen pair of integers i and j.

Now, if possible, suppose w is a second common fixed point of P_i and S_i. Then

$$\begin{aligned} d(w, z) &= d(P_i w, Q_j z) \\ &< \max\{d(S_i w, T_j z), d(P_i w, S_i w), d(Q_j z, T_j z)\} \\ &= d(w, z), \end{aligned}$$

a contradiction. Hence z is the unique common fixed point of P_i and S_i. Similarly z is the unique common fixed point of Q_j and T_j.

Now, if we keep i fixed and vary j, we find that z is the unique common fixed point of Q_j and T_j for every value of j. On the other hand, since $Q_j z = z = T_j z$, if we keep j fixed and vary i and also drop the assumption of continuity of S_i we can easily show that z is the unique common fixed point of P_i and S_i for every value of i.

Using similar type of arguments we can prove that z is the unique common fixed point of P_i, Q_j, S_i, and T_j if P_i is assumed to be continuous instead of S_i.

Similarly z is the unique common fixed point of P_i, Q_j, S_i and T_j if either Q_j or T_j is continuous instead of S_i or P_i respectively.

This completes the proof of the theorem.

Remark 1. In the above theorem if we take

$P_i = P$, $Q_j = Q$, $S_i = AS$, and $T_j = T$ for each i and j, we obtain the theorem of Pant [2].

Remark 2. In the above theorem if we take $P_i = P$, $Q_j = Q$, $S_i = S$ and $T_j = T$ for each i and j and let $h(\varepsilon) = 2(1 - c)\,\varepsilon/c$, $0 < c < 1$, then we obtain Theorem 2 of Fisher [1].

Remark 3. In the above theorem if we take $P_i = Q_j = g$ and $S_i = T_j = f$ for every value of i and j and let

$$\max \{d(S_i x, T_j y), d(P_i x, S_i x), d(Q_j y, T_j y)\} = d(S_i x, T_j y),$$

then we can drop the assumption on nondecreasing character of $h(\varepsilon)$ and we obtain the Theorem 2.4 of Park and Bae [3].

We now give an example of sequences of mappings satisfying the conditions of the above theorem and having a unique common fixed point.

Example. Let $X = [0, 1]$ and let d be the usual metric on X. Define selfmappings P_i, Q_j, S_i and T_j on X, $i, j = 1, 2, 3, \ldots$, as follows

$P_i x = ix/(6i + 1)$, $\quad x \neq 1$, $\quad P_i 1 = i/(12i + 1)$

$Q_j x = 0$, $\quad x \neq 1$, $\quad Q_j 1 = j/(6j + 1)$

$S_i x = ix/(2i + 1)$, $\quad x \neq 1$, $\quad S_i 1 = i/(4i + 1)$

$T_j x = x$ for each x in X.

Then $\{P_i\}$, $\{Q_j\}$, $\{S_i\}$ and $\{T_j\}$ satisfy all the conditions of the theorem and have a unique common fixed point $x = 0$.

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