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COMMON FIXED POINTS OF WEAKLY COMMUTING MAPPINGS

By

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ABSTRACT

A common fixed point theorem for two sequences of self-mappings, respectively weakly commuting with two given self-mapping of a complete metric space and satisfying a Meir and Keeler type contractive condition is obtained.The existence of the fixed point has been established assuming the continuity of only one of the mappings. Our work generalizes several well known results on contractive mappings.

1. INTRODUCTION

The study of common fixed points of contractive type mappings has emerged as an area of vigorous research activity and a number of interesting results have been reported.Majority of these results either deal with commuting mappings or with more generalized concept of weak commutativity of mappings introduced by Sessa [8]. Jungeck [2] introduced the notion of compatibility of mappings, also called asymptotic commutativity by Tivari and Singh[9] in an independent formulation. It was claimed that weak commutativity implies compatibility [3], [9] but not conversely [7]. However, in a review of [3] (Mathematical Review 89 h : 54030) Singh has shown the existence of a weakly commuting pair of mappings satisfying a contractive condition for which there exists no sequence of points satisfying the condition of compatibility.

In this paper we obtain a common fixed point theorem for two sequences of selfmappings respectively weakly commuting with two selfmappings and satisfying a Meir and Keeler type contractive condition. The theorem assumes the continuity of only one of the mappings. The mapping condition studied by us is a generalization of the mapping condition 22 of Rhoades [6]. Our work generalizes the results due to Fisher [1], Pant [4], Park and Rhoades [5] and a number of other results.

2. RESULTS

If (X, d) be a metric space, two selfmappings F and G of X are called weakly commuting provided $d(FGx, GFx) \leq d(Fx, Gx)$ for each x in X.

Theorem. Let $\{P_i\}$ and $\{Q_j\}$, $i, j = 1, 2, 3, \ldots$, be sequences of selfmappings and let S and T be selfmappings of a complete metric space (X, d) satisfying the conditions :

Given $\varepsilon > 0$, there exists an $h(\varepsilon) > 0$, $h(\varepsilon)$ being a nondecreasing function of ε, such that for all x, y in X

$$\varepsilon \leq \max \{d(Sx, Ty), d(P_i x, Sx), d(Q_j y, Ty), [d(P_i x, Ty) + d(Q_j, Sx)]/2\} < \varepsilon + h$$

$$\Rightarrow \quad d(P_i x, Q_j y) < \varepsilon, \qquad \ldots (1)$$

$$P_i x = Q_j y \quad \text{whenever } P_i x = Sx,\ Q_j y = Ty. \qquad \ldots (2)$$

Let the range of T contain the range of each P_i and the range of S contain the range of each Q_j. If each P_i weakly commutes with S and each Q_j weakly commutes with T and if one of the mappings $\{P_i\}, \{Q_j\}, S$ or T be continuous then the mappings $\{P_i\}, \{Q_j\}, S$ and $T, i, j = 1, 2, 3, \ldots$, have a unique common fixed point which is also the unique common fixed point of P_i and S and of Q_j and T.

Proof. First, with the help of (1), we note that for all x, y in X such that $P_i x \neq Sx$, $Q_j y \neq Ty$, $i, j = 1, 2, 3, \ldots$,

$$d(P_i x, Q_j y) < \max\{d(Sx, Ty), d(P_i x, Sx), d(Q_j y, Ty), [d(P_i x, Ty) + d(Q_j y, Sx)]/2\}. \qquad \ldots (3)$$

Secondly, the nondecreasing character of $h(\varepsilon)$ implies that given $\varepsilon > 0$, there exists $\varepsilon_0 > 0$, such that $\varepsilon_0 < \varepsilon < \varepsilon_0 + h(\varepsilon_0)$ or equivalently

$$\max \{d(Sx, Ty), d(P_i x, Sx), d(Q_j y, Ty), [d(P_i x, Ty) + d(Q_j y, Sx)]/2\} = \varepsilon$$

$$\rightarrow \quad d(P_i x, Q_j y) < \varepsilon_0,\ \varepsilon_0 < \varepsilon. \qquad \ldots (4)$$

Let us arbitrarily select a pair of integers i and j and let x_0 be any point in X. Choose a sequence of points $\{x_n : n = 0, 1, 2, \ldots\}$ in X defined by $P_i x_{2n} = Tx_{2n+1}$ and $Q_j x_{2n+1} = Sx_{2n+2}$. This can be done since the ranges of T and S respectively contain the ranges of P_i and Q_j. We can assume that $P_i x_{2n} \neq Q_j x_{2n+1}$ and $Q_j x_{2n+1} \neq P_i x_{2n+2}$ for every value of n, otherwise the existence of the fixed point is easy to establish. Then from (3) we obtain

$$d(P_i x_{2n}, Q_j x_{2n+1}) < d(Q_j x_{2n-1}, P_i x_{2n}) \qquad \ldots (5)$$

and $d(Q_j x_{2n-1}, P_i x_{2n}) < d(P_i x_{2n-2}, Q_j x_{2n-1})$. ... (6)

Similarly, for every integer $p > 0$

$$d(Q_j x_{2n+1}, P_i x_{2(n+p)+2}) < d(P_i x_{2n}, Q_j x_{2(n+p)+1}) + d(P_i x_{2n}, Q_j x_{2n+1}) \quad \text{... (7)}$$

and $$d(P_i x_{2n}, Q_j x_{2(n+p)+1}) < d(Q_j x_{2n-1}, P_i x_{2(n+p)}) + d(Q_j x_{2n-1}, P_i x_{2n}). \quad \text{... (8)}$$

In view of (5) and (6) we claim that $\lim_{n\to\infty} d(P_i x_{2n}, Q_j x_{2n+1}) = 0 = \lim_{n\to\infty} d(Q_j x_{2n+1}, P_i x_{2n+2})$. For, if not, suppose for instance $\lim_{n\to\infty} d(P_i x_{2n}, Q_j x_{2n+1}) = r$, $r > 0$. Then given $h > 0$, there exists a positive integer N_1, such that for each integer $m \geq N_1$ we have

$$r \leq d(P_i x_{2m}, Q_j x_{2m+1}) < r + h \quad \text{... (9)}$$

or $$r \leq \max\{d(Sx_{2m+2}, Tx_{2m+1}), d(P_i x_{2m+2}, Sx_{2m+2}), d(Q_j x_{2m+1}, Tx_{2m+1}), [d(P_i x_{2m+2}, Tx_{2m+1}) + d(Q_j x_{2m+1}, Sx_{2m+2})/2\} < r + h. \quad \text{... (10)}$$

Selecting h in (10) in accordance with (1), for each integer $m \geq N_1$, we obtain $d(P_i x_{2m+2}, Q_j x_{2m+1}) < r$ and so $d(P_i x_{2m+2}, Q_j x_{2m+3}) < r$, which contradicts (9). Therefore

$$\lim_{n\to\infty} d(P_i x_{2n}, Q_j x_{2n+1}) = 0 = \lim_{n\to\infty} d(Q_j x_{2n+1}, P_i x_{2n+2}). \quad \text{... (11)}$$

Also by virtue of inequalities (7), (8) and equation (11), it follows that

$$\lim_{n\to\infty} d(P_i x_{2n}, Q_j x_{2(n+p)+1}) = \lim_{n\to\infty} d(Q_j x_{2n+1}, P_i x_{2(n+p)+2}).$$

Now,if possible, suppose $\lim_{n\to\infty} d(P_i x_{2n}, Q_j x_{2(n+p)+1}) = r$, $r > 0$.

Then in view of this and equation (11), given $h > 0$ there exists a positive integer N_2 such that for each integer $m \geq N_2$, we have

$$r \leq d(P_i x_{2m}, Q_j x_{2(m+p)+1}) = d(Tx_{2m+1}, Sx_{2(m+p)+2}) < r + h/4, \quad \text{... (12)}$$

$$r \leq d(Q_j x_{2m+1}, P_i x_{2(m+p)+2}) = d(Sx_{2m+2}, Tx_{2(m+p)+3}) < r + h/4, \quad \text{... (13)}$$

$$d(P_i x_{2m}, Q_j x_{2m+1}) < h/4,$$

and $\quad d(Q_j x_{2m-1}, P_i x_{2m}) < h/4.$

Therefore for each $m \geq N_2$ we have

$$r \leq \max\{d(Sx_{2m+2}, Tx_{2(m+p)+3}), d(P_i x_{2m+2}, Sx_{2m+2}), d(Q_j x_{2(m+p)+3}, Tx_{2(m+p)+3}), [d(P_i x_{2m+2}, Tx_{2(m+p)+3}) + d(Q_j x_{2(m+p)+3}, Sx_{2m+2})]/2\} < r + h/4. \quad \ldots (14)$$

Selecting h in (14) in accordance with (1), for each $m \geq N_2$, we have $d(P_i x_{2m+2}, Q_j x_{2(m+p)+3}) < r$, which contradicts (12). Hence $\{P_i x_0, Q_j x_1, \ldots, P_i x_{2n}, Q_j x_{2n+1}, \ldots\}$ is a Cauchy sequence in the complete metric space X and so has a limit point z in X. Also the sequences $\{P_i x_{2n}\} = \{Tx_{2n+1}\}$ and $\{Q_j x_{2n+1}\} = \{Sx_{2n+2}\}$ converge to the point z.

Let us now suppose that the mapping S is continuous. Then since P_i and S commute weakly, the sequences $\{P_i Sx_{2n}\}$ and $\{SSx_{2n}\}$ converge to Sz. We claim that $z = Sz$. For if $z \neq Sz$, then the inequality

$$d(P_i Sx_{2n}, Q_j x_{2n+1}) < \max\{d(SSx_{2n}, Tx_{2n+1}), d(P_i Sx_{2n}, SSx_{2n}), d(Q_j x_{2n+1}, Tx_{2n+1}), [d(P_i Sx_{2n}, Tx_{2n+1}) + d(Q_j x_{2n+1}, SSx_{2n})]/2\}$$

on letting $n \to \infty$ and in view of (4) leads to $d(z, Sz) < d(z, Sz)$, contradiction. Therefore $z = Sz$. Similarly the inequality

$$d(P_i z, Q_j x_{2n+1}) < \max\{d(Sz, Tx_{2n+1}), d(P_i z, Sz), d(Q_j x_{2n+1}, Tx_{2n+1}), [d(P_i z, Tx_{2n+1}) + d(Q_j x_{2n+1}, Sz)]/2\}$$

yields $z = P_i z$. This means that there exists a point z_0 *in* X such that $Sz = z = P_i z = Tz_0$, as the range of P_i is contained in the range of T. Morevoer, the inequality.

$$d(P_i z, Q_j z_0) \ \max\{d(Sz, Tz_0), d(P_i z, Sz), d(Q_j z_0, Tz_0), [d(P_i z, Tz_0) + d(Q_j z_0, Sz)]/2\}$$

yields $z = Q_j z_0 = Tz_0$. This equation, in view of weak commutativity of Q_j and T, implies $Q_j z = Tz$. Finally $Q_j z = Tz$ together with (3) leads to $z = Q_j z = Tz$.

We have therefore proved that z is a common fixed point of P_i, Q_j, S and T for an arbitrarily chosen pair of integers i and j.

Now, if possible suppose w is a second common fixed point of P_i and S. Then

$$d(w, z) = d(P_i w, Q_j z) < \max\{d(Sw, Tz), d(P_i w, Sw), d(Q_j z, Tz),$$

$$[d(P_i w, Tz) + d(Q_j z, Sw)]/2\}$$
$$= d(w, z),$$

a contradiction. Hence z is the unique common fixed point of P_i and S. Similarly z is also the unique common fixed point of Q_j and T.

Now if we keep i fixed and vary j, we find that z is the unique common fixed point of Q_j and T for every value of j. On the other hand, since $z = Q_j z = Tz$, if we keep j fixed and vary i it can be easily shown that z is the unique common fixed point of P_i and S for every value of i.

Using similar type of arguments we can prove that z is the unique common fixed point of P_i, Q_j, S and T if. P_i is assumed to be continuous, for some value of i, instead of S.

Similarly z is the unique common fixed point of P_i, Q_j, S and T if either T or Q_j is continuous instead of S or P_i. respectively.

This completes the proof of the theorem.

Remark 1. In the above theorem if we take $P_i = P = Q_j$ for each i and each j and $S = T$, we obtain Theorem 4 of Park and Rhoades [5] as a particular case of our theorem. In that case we can drop the assumption on nondecreasing character of $h(\varepsilon)$

Remark 2. In this theorem if we take $P_i = P$ and $Q_j = Q$ for each i and each j and if we take $\max\{d(Sx, Ty, d(Px, Sx), d(Qy, Ty), [d(Px, Ty) + d(Qy, Sx]/2\} = \max \{d(Sx, Ty), d(Px, Sx), d(Qy, Ty)\}$, we obtain the Theorem of Pant [4]. Further by choosing $h(\varepsilon) = 2(1 - c)\, \varepsilon/c$, $0 < c < 1$, we shall obtain Theorem 2 of Fisher [1] as a special case.

We now give an example of mappings satisfying the conditions of the above theorem and having a unique common fixed point.

Example. Let $X = [0, 1]$ and let d be the usual metric on X. Define selfmappings P_i, Q_j, S and T on $X, i, j = 1, 2, 3, \ldots$, as follows

$P_i x = ix/(6i + 1)$,	$x \neq 1$,	$P_i 1 = i/(12i + 1)$
$Q_j x = 0$,	$x \neq 1$,	$Q_j 1, = j/(6j + 1)$
$Sx = x/2$,	$x \neq 1$,	$S1 = 1/4$

$Tx = x$ for each x in X.

Then $\{P_i\}$, $\{Q_j\}$, S and T satisfy all the conditions of the theorem and have a unique common fixed point $x = 0$.

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