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(Dedicated to Professor J.N. Kapur on his 70th Birthday)

CERTAIN GENERATING FUNCTIONS FOR THE KONHAUSER'S POLYNOMIALS $Z_n^{\alpha}(x; k)$

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ABSTRACT

In this paper, we derive some generating relations for one set $Z_n^{\alpha}(x; k)$ of the Konhauser biorthogonal polynomials. For these polynomials we have established some generating relations involving Kampe de Feriet's double hypergeometric function and Srivastava's general class of triple hypergeometric functions. These generating relations are deduced from certain well known results involving generalized hypergeometric polynomials.

1. INTRODUCTION

Konhauser [3] discussed two polynomial sets $Y_n^{\alpha}(x; k)$ and $Z_n^{\alpha}(x; k)$ which are biorthogonal with respect to the weight function $(x)^{\alpha}e^{-x}$ over the interval $(0, \infty)$, where $\alpha > -1$ and k is a positive integer $Y_n^{\alpha}(x; k)$ is a polynomial of degree n in x^k.

The biorthogonality relation is given by [3,p.303]

$$\int_0^{\infty} x^{\alpha}e^{-x}\, Y_n^{\alpha}(x; k)\, Z_m^{\alpha}(x;k)\, dx = \frac{\Gamma(km + \alpha + 1)}{m\,!}\, \delta_{nm}, \qquad \text{... (1.1)}$$

where δ_{nm} is the Kronecker delta.

For $k = 1$, these polynomials reduce to the Laguerre polynomials $L_n^{(\alpha)}(x)$ and for $k = 2$, *these polynomials were studied by Spencer and Fano [9] in certain calculaions involving the penetration of gamma rays through matter and were subsequently discussed by Preiser [7].*

These polynomials were further investigated by Prabhakar [5,6], Srivastava [11] to [15], Patil and Thakare [4], Agarwal and Manocha [1] and, Srivastava and Singh [10].

One set $Z_n^{\alpha}(x; k)$ of the Konhauser's biorthogonal polynomials is given explicitly by [3].

$$Z_n^{\alpha}(x;k) = \frac{\Gamma(kn+\alpha+1)}{n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj+\alpha+1)} \quad \ldots (1.2)$$

$\alpha > -1$ and k is a positive integer. An immediate consequence of (1.2) is the formula

$$Z_n^{\alpha}(x;k) = \frac{(\alpha+1)_{kn}}{n\,!} \cdot {}_1F_k \left[\begin{matrix} -n; \\ (\Delta\, k; \alpha+1); \end{matrix} \; (x/k)^k \right] \quad \ldots (1.3)$$

where $\Delta(\lambda, \mu)$ represents the array of parametes $\frac{\mu}{\lambda}, \frac{\mu+1}{\lambda}, \ldots, \frac{\mu+\lambda-1}{\lambda}$.

The Kampe de Feriet's double hypergeometric function (in the contracted notation of Burchnall and Chaundy [2, p. 112] is defined by

$$F^{(2)} \left[\begin{matrix} (a) : (b) : (c) : \\ (e) : (g) : (h) : \end{matrix} \; x, y \right] = \sum_{m,n=0}^{\infty} \frac{[(a)]_{m+n} [(b)]_m [(c)]_n \, x^m y^n}{[(e)]_{m+n} [(g)]_m [(h)]_n \, m\,!\, n\,!} \quad \ldots (1.4)$$

and a general class of triple hypergeometric functions, due to H.M.Srivastava, is defined by [11, p. 428]

$$F^{(3)} \left[\begin{matrix} (a) :: (b); (b') ; (b'') : (c) ; (c') ; (c'') ; \\ (e) :: (g) ; (g') ; (g'') : (h) ; (h') ; (h''); \end{matrix} \; x, y, z \right]$$

$$= \sum_{l,m,n=0}^{\infty} \frac{[(a)]_{l+m+n} \, [(b)]_{l+m} [(b')]_{m+n} [(b'')]_{n+l}}{[(e)]_{l+m+n} \, [(g)]_{l+m} [(g')]_{m+n} [(g''))]_{n+l}} ,$$

$$\frac{[(c)]_l \, [(c')]_m \, [(c'')]_n}{[(h)]_l [(h')]_m [(h'')]_n} \cdot \frac{x^l}{l\,!} \frac{y^m}{m\,!} \frac{z^n}{n\,!} . \quad \ldots (1.5)$$

In the definitions of (1.4) and (1.5),as well as in what follows, (a) and $(b)_n$ abbreviate the sequence of A parameters $a_1, \ldots, a_A$ and the product $(a_1)_n, \ldots, (a_A)_n$. respectively.

Here, we deduce some generating relations for the polynomials $Z_n^{\alpha}(x;k)$ from some well known results for certain generalized hypergeometric polynomials (if, e.g.) Srivastava and Manocha (16) involving the function $F^{(2)}\,(x,y)$ and $F^{(3)}\,(x,y,z)$. All the generating relations, would reduce, when $k = 1$, to known results for Laguerre polynomials.

2. GENERATING RELATIONS

The generating relations that we deduce here from certain well known results are :

$$\sum_{n=0}^{\infty} \frac{[(a)]_n}{[(e)]_n (\alpha+1)_{kn}} Z_n^{\alpha}(x;k) y^n$$

$$= F^{(2)} \left[\begin{matrix} (a): & ;-; \\ (e):(\alpha+1)/k, ..., (\alpha+k)/k; -; \end{matrix} \; y, -\{(x)^k/k \,.\, y\} \right] \quad ... (2.1)$$

$$\sum_{n=0}^{\infty} \frac{[(a)]_n \left(1-\frac{\alpha+1}{k}\right)_n \cdots \left(1-\frac{\alpha+k}{k}\right)_n}{[(e)]_n (\alpha+1-kn)_{kn}} Z_n^{\alpha-kn}(x;k)\, y^n$$

$$= F^{(2)} \left[\begin{matrix} (a); -; 1-(\alpha+1)/k, ..., 1-(\alpha+k)/k; \\ (e): -; \text{———}; \end{matrix} \; y, -\{(x/k)^k \,.\, y\} \right] \quad ... (2.2)$$

$$= \sum_{n=0}^{\infty} \frac{[(a)]_m [(b'')]_n}{[(e)]_n [(g'')]_n (\alpha+1)_{kn}} {}_{A+C'}F_{E+H'} \left[\begin{matrix} (a)+n, (c') \\ (e)+n, (h'); \end{matrix} \; y \right] . Z_n^{\alpha}(x;k) z^n$$

$$= F^{(3)} \left[\begin{matrix} (a) :: -; (b''): \text{———}; (c); -; \\ (e) :: -; -; (g''): (\alpha+1)/k, ..., (\alpha+k)/k; (h'); -; \end{matrix} \; y, z, -\{(x/k)^k \,.\, z\} \right] \quad ... (2.3)$$

and

$$\sum_{n=0}^{\infty} \frac{[(a)]_n [(b'')]_n \left(1-\frac{\alpha+1}{k}\right)_n \cdots \left(1-\frac{\alpha+k}{k}\right)_n}{[(e)_n [(g'')]_n (\alpha+1-kn)_{kn}} .$$

$${}_{A+E'}F_{E+H'} \left[\begin{matrix} (a)+n, (c'); \\ (e)+n, (h'); \end{matrix} \; y \right] . Z_n^{\alpha-kn}(x;k) z^n$$

$$F^{(3)} \left[\begin{matrix} (a) :: -; -; (b''): -; (c'); 1-(\alpha+1)/k, ... 1-(\alpha+k)/k; \\ (e) :: -; -; (g'') -; (h'); \text{———}; \end{matrix} \; x, z, -\{(x/k)^k \,.\, z)\} \right] \quad ... (2.4)$$

Derivations of (2.1) and (2.2) : From Srivastava and Manocha (16,p.194), and Rainville [18,p. 32], we have

$$\sum_{n=0}^{\infty} \frac{[(a)]_n\, [(c)]_n}{[(e)]_n\, [(h)]_n}\; {}_{H+B+1}F_{C+G}\left[\begin{matrix} -n,\, 1-(h)-n,\, (b): \\ 1-(c)-n,\, (g); \end{matrix}\; x\right] \frac{y^n}{n\,!}$$

$$= F^{(2)}\left[\begin{matrix} (a) : (b)\,;\, (c); \\ \\ (e) : (g);\, (h); \end{matrix}\; y,\, [(-1)]^{c-h-1}\,.\, xy\right] \qquad \ldots (2.5)$$

Now the result (2.1) would follow at once if we interpret (2.5) in the light of (1.3).

In deducing (2.2), from the known result (2.5) we make use of

$$Z_n^{\alpha - kn}(x;\, k) = \frac{(\alpha + 1 - kn)_{kn}}{n\,!}\; {}_1F_k\left[\begin{matrix} -n; \\ \Delta(k;\, \alpha + 1 - kn); \end{matrix}\; (x/k)^k\right].$$

Derivations of (2.3) and (2.4) :

In case of the function $F^{(3)}(x, y, z)$, we recall from Srivastava and Manocha [16,p.157] that

$$\sum_{n=0}^{\infty} \frac{[(a)]_n\, [(b'')]_n\, [(c'')]_n}{[(e)]_m\, [(g'')]_n\, [(h'')]_n}\; {}_{A+C'}F_{E+H'}\left[\begin{matrix} (a)+n,\, (c'); \\ (e)+n,\, (h'); \end{matrix}\; y\right]$$

$$.\; {}_{C+H''+1}F_{H+C''}\left[\begin{matrix} -n,\, (c),\, 1-(h'')-n; \\ (h),\, 1-(c'')-n; \end{matrix}\; x\right] \frac{z^n}{n\,!}$$

$$= F^{(3)}\left[\begin{matrix} (a) :: -;\, -;\, (b'') : (c);\, (c');\, (c''); \\ (e) :: -;\, -;\, (g'') : (h);\, (h');\, (h''); \end{matrix}\; y,\, z,\, xz\right] \qquad \ldots (2.6)$$

The specialised forms of (2.6) lead us to the desired generating relations (2.3) and (2.4).

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