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AN OPERATIONAL CALCULUS FOR THE INDEX $_2F_1$ – TRANSFORM

By

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ABSTRACT

In this paper an operational calculus for the index $_2F_1$ – transform is developed. Furthermore, this calculus is applied to solve certain type of differential equations involving generalized functions.

1. Introduction

The index $_2F_1$-transform for real-valued functions is defined in [3] (see also [4], by the formula:

$$F(\tau) = \int_0^\infty \boldsymbol{F}(\mu, \alpha, \tau, x) f(x) dx, \qquad \text{... (1.1)}$$

where

$$\boldsymbol{F}(\mu, \alpha\, \tau, x) = {}_2F_1(\mu + \tfrac{1}{2} + i\tau, \mu + \tfrac{1}{2} - i\tau; \mu + 1; -x)x^{\alpha}, \qquad \text{... (1.2)}$$

$_2F_1$ being the Gauss hypergeometric function. Here α and μ are complex parameters and τ is a positive real parameter.

This integral transform was extended to certain spaces of generalized functions (see [5]). For it, a testing function space (namely, $U_{\alpha, \mu, \alpha}$) was introduced in the following manner: a smooth function $\phi(x)$ on $0 < x < \infty$, belongs to $U_{a, \mu, \alpha}$ if and only if

$$\gamma_{k, a, \mu, \alpha}(\phi) = \sup_{0 < x < \infty} \left| (2x+1)^{a} x^{\mu/2 - \alpha} (x+1)^{\mu/2} A_x^k \phi(x) \right| < \infty \qquad \text{... (1.3)}$$

with $a \in [0, \frac{1}{2})$, $\mu, \alpha \in C$, $k = 0, 1, 2, ...$ and where A_x denotes the differential operator :

$$A_x := x^{\alpha - \mu}(x+1)^{-\mu} D_x\, x^{\mu + 1} (x+1)^{\mu + 1} D_x\, x^{-\alpha}. \qquad \text{... (1.4)}$$

The topology in $U_{\alpha, \mu, \alpha}$ is induced by the countable family of seminorms $\{\gamma_{k, a, \mu, \alpha}\}$. It turns out that the kernel function $\boldsymbol{F}(\mu, \alpha, \tau, x)$ is in $U_{\alpha, \mu, \alpha}$. As it is usual, the dual space of $U_{\alpha, \mu, \alpha}$ is denoted by $U'_{a, \mu, \alpha}$, and I denotes the real interval $(0, \infty)$.

The transform $F(\tau)$ of the generalized function $f \in U'_{a,\mu,\alpha}$ is then defined as follows :

$$ {}_2\mathcal{F}_1(f) = F(\tau) = \langle f(x), \boldsymbol{F}(\mu, \alpha, \tau, x) \rangle, \quad \tau > 0 \qquad \dots (1.5)$$

and for $a \in [0, \frac{1}{2})$ and $f \in \mathcal{E}'(\boldsymbol{I})$ the following inversion formula holds :

$$\langle f, \phi \rangle = \lim_{N \to \infty} \left\langle \int_0^N S(\mu, \tau)\, \boldsymbol{G}(\mu, \alpha, \tau, x)\, F(\tau) d\tau, \phi(x) \right\rangle \qquad \dots (1.6)$$

with Re $\alpha > 0$, Re $\mu > 0$, $\frac{1}{8} <$ Re $(\mu - \alpha) < \frac{1}{4}$ and Re $(\frac{\mu}{2} - \alpha) < -\frac{1}{2}$. Here,

$$S(\mu, \tau) = \frac{2}{\pi \Gamma(\mu + 1)^2}\, \tau\, sh\, \pi \tau\, \Gamma(\mu + \tfrac{1}{2} + i\tau)\, \Gamma(\mu + \tfrac{1}{2} - i\tau)$$

and

$$\mathbf{G}(\mu, \alpha, \tau, x) = x^{\mu - \alpha}\, {}_2F_1(\tfrac{1}{2} + i\tau, \tfrac{1}{2} - i\tau; \mu + 1; -x).$$

Further more, the following uniqueness theorem was proved : if $f, g \in \mathcal{E}'(\boldsymbol{I})$ and ${}_2\mathcal{F}_1[f] = {}_2\mathcal{F}_1[g]$, then $f = g$.

In [5] was also established that for every $f \in U'_{a,\mu,\alpha}$ there exists a non- negative integer τ such that

$$ {}_2\mathcal{F}_1(f) = F(\tau) = O\left(\tau^{2r - \text{Re}\,\mu - \frac{1}{2}}\right), \quad \tau \to \infty \qquad \dots (1.7)$$

On the other hand, the following relation holds :

$$ {}_2\mathcal{F}_1\left((A_x')^{\kappa} f\right) = (-1)^{\kappa} \left[\left(\mu + \tfrac{1}{2}\right)^2 + \tau^2\right]^{\kappa} {}_2\mathcal{F}_1(f), \quad \kappa \in N. \qquad \dots (1.8)$$

A'_x being the adjoint operator of A_x defined by (1.4).

In this paper we consider the operational equation :

$$P(A'_x) u = g,$$

where $g \in \mathcal{E}'(\boldsymbol{I})$, P is any polynomial different from zero in $(-\infty, 0)$ and $\alpha, \mu \in \boldsymbol{R}$.

Our aim is to find a generalized function $u \in \mathcal{E}'(\boldsymbol{I})$ satisfying the above operational equation. Spaces $\mathcal{D}(I)$ and $\mathcal{E}(I)$ and their duals $D'(I)$ and $\mathcal{E}'(I)$have its usual meaning [7].

2. The operational equation

Let us consider the following equation :

$$P(A'_x) u = g, \qquad \dots (2.1)$$

where $g \in \mathcal{E}'(\boldsymbol{I})$, $P(z)$ denotes an arbitrary polynomial without zeros in $-\infty < z \leq 0$ and just as in (1.8) A_x' is the differential operator

$$A_x' = x^{-\alpha} D_x x^{\mu + 1} (x + 1)^{\mu + 1} D_x x^{\alpha - \mu} (x + 1)^{-\mu},$$

which is the adjoint of A_x.

By applying the generalized index ${}_2\mathcal{F}_1$-transform to both sides of (2.1) and using (1.8),we get

$$P\left(-\left(\mu+\tfrac{1}{2}\right)^2-\tau^2\right)U(\tau)=G(\tau),$$

where $U(\tau)$ and $G(\tau)$ represent the generalized index ${}_2\mathcal{F}_1$-tranform of $u(x)$ and $g(x)$, respectively. Now, since $P\left(-\left(\mu+\tfrac{1}{2}\right)^2-\tau^2\right)\neq 0$,by means of the inversion formula (1.6), and for any $\phi\in\mathcal{D}(I)$ it is found that

$$\langle u,\phi\rangle=\lim_{N\to\infty}\left\langle\int_0^N S(\mu,\tau)\,G(\mu,\alpha,\tau,x)\frac{G(\tau)}{P\left(-\left(\mu+\tfrac{1}{2}\right)^2-\tau^2\right)}d\tau,\phi(x)\right\rangle \qquad \text{... (2.2)}$$

Thus, we have obtained formally a solution of (2.1). Now we must prove that (2.2) is certainly a generalized function and that it satisfies the equation (2.1). For this, the following Lemma is required.

Lemma 2.1 *For $x\in[a,b]$ and $\mu>0$, there exists a $T_1>0$ such that $\forall\,\tau\le T_1$*

$$\left|{}_2F_1\left(\tfrac{1}{2}+i\tau,\tfrac{1}{2}-i\tau\,;\mu+1;-x\right)\right|\le B_1$$

and there also exists a $T_2>0$, such that $\forall\,\tau\ge T_2$

$$\left|{}_2F_1\left(\tfrac{1}{2}+i\tau,\tfrac{1}{2}-i\tau\,;\mu+1;-x\right)\right|\le B_2\,\tau^{-\frac{1}{2}}$$

with $B_1>0$ *and* $B_2>0$.

Proof. Starting from the integral representation [6] (p. 248).

$${}_2F_1(\alpha,\beta;\gamma;-x)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\frac{4}{x^{\alpha+\beta/2}}\int_0^\infty K_{\alpha-\beta}\left(\frac{2s}{\sqrt{x}}\right)J_{\gamma-1}(2s)s^{\alpha+\beta-\gamma}\,ds.$$

which is valid for $\operatorname{Re}\alpha>0$, $\operatorname{Re}\beta>0$, and K, J being the well-known Bessel functions, we can write :

$${}_2F_1\left(\tfrac{1}{2}+i\tau,\tfrac{1}{2}-i\tau;\mu+1,-x\right)=\frac{\Gamma(\mu+1)}{\Gamma(\frac{1}{2}+i\tau)\Gamma(\frac{1}{2}-i\tau)}4x^{-1/2}\int_0^\infty K_{2\tau i}\left(\frac{2s}{\sqrt{x}}\right)J_\mu(2s)s^{-\mu}ds$$

Hence, for a suitable $T_1>0$, if $\tau\le T_1$ one has

$$\left|K_{2\tau i}\left(\frac{2s}{\sqrt{x}}\right)\right|\le K_0\left(\frac{2s}{\sqrt{x}}\right).$$

Thus, for $\tau\le T_1$,

$$\left|{}_2F_1\left(\tfrac{1}{2}+i\tau,\tfrac{1}{2}-i\tau;\mu+1;-x\right|\le B_1$$

with $B_1>0$.

On the other hand, for $\tau\to\infty$,

$$\left|K_{2\tau i}\left(\frac{2s}{\sqrt{x}}\right)\right| \le M_1\, e^{-\pi\tau}\, \tau^{-\frac{1}{2}}$$

holds for certain $M_1 > 0$ (see [2] 7.14.2 (69)).

Moreover, $\Gamma\left(\frac{1}{2} + i\tau\right)\Gamma\left(\frac{1}{2} - i\tau\right) = \frac{\pi}{ch\,\pi\,\tau}$.

Then for $\tau \ge T_2$:

$$\left|{}_2F_1\left(\tfrac{1}{2} + i\tau, \tfrac{1}{2} - i\tau; \mu + 1; -x\right)\right| \le M_2 \tau^{-1/2} e^{-\pi\tau}\, ch\pi\tau \int_0^\infty s^{-\mu}\, |J_\mu(2s)|\, ds \le B_2 \tau^{-\frac{1}{2}}$$

B_2 being a positive constant □

Now, if we again consider the equation (2.1) it results, after taking a polynomial $Q(z)$ of degree $r + \mu + 1$ without zeros on $-\infty < z \le 0$, r being the integer given in (1.7), that the convergence of the right- hand side of (2.2) can be established as follows. First, we have

$$\left\langle \int_0^N S(\mu,\tau)\, \boldsymbol{G}(\mu, \alpha, \tau, x) \frac{G(\tau)}{P(-(\mu + \frac{1}{2})^2 - \tau^2)}\, d\tau,\ \phi(x) \right\rangle =$$

$$\left\langle Q(A'_x) \int_0^N \frac{S(\mu, \tau)\boldsymbol{G}(\mu, \alpha, \tau, x) G(\tau)}{P(-(\mu + \frac{1}{2})^2 - \tau^2)\, Q(-(\mu + \frac{1}{2})^2 - \tau^2)}\, d\tau,\ \phi(x) \right\rangle =$$

$$\left\langle \int_0^N \frac{S(\mu, \tau)\, \boldsymbol{G}(\mu, \mu, \tau, x)\, G(\tau)}{P(-(\mu + \frac{1}{2})^2 - \tau^2)\, Q(-(\mu + \frac{1}{2})^2 - \tau^2)}\, d\tau,\ Q(A_x)\, \phi(x) \right\rangle .$$

This follows by making use of :

$$A'_x\, \boldsymbol{G}(\mu, \alpha, \tau, x) = -\left[(\mu + \tfrac{1}{2})^2 + \tau^2\right] \boldsymbol{G}(\mu, \alpha, \tau, x).$$

Thus, if the support of ϕ is contained in $[a, b]$, the expression (2.2) can be written as

$$\lim_{N \to \infty} \int_0^N \frac{S(\mu, \tau) G(\tau)}{P(-(\mu + \frac{1}{2})^2 - \tau^2)\, Q(-(\mu + \frac{1}{2})^2 - \tau^2)}\, d\tau \int_a^b \boldsymbol{G}(\mu, \alpha, \tau, x)\, \phi(x)\, dx \qquad \text{... (2.3)}$$

Then, by invoking Lemma 2.1 we can find suitable constants C, D, E, N_1 and N_2 such that

$$\int_0^N \left| \frac{S(\mu, \tau) G(\tau)}{P(-(\mu + \frac{1}{2})^2 - \tau^2)\, Q(-(\mu + \frac{1}{2})^2 - \tau^2)} \right| d\tau \int_a^b |\boldsymbol{G}(\mu, \alpha, \tau, x)\, \phi(x)|\, dx \le$$

$$\leq C\int_0^{N_1}\left|\frac{S(\mu,\tau)}{P\,(-(\mu+\frac{1}{2})^2-\tau^2)\,Q\,(-(\mu+\frac{1}{2})^2-\tau^2)}\right|d\tau+$$

$$+D\int_{N_1}^{N_2}\left|\frac{G(\tau)S(\mu,\tau)}{P\,(-(\mu+\frac{1}{2})^2-\tau^2)\,Q\,(-(\mu+\frac{1}{2})^2-\tau^2)}\right|d\tau.$$

$$\int_a^b |{}_2F_1\,(\frac{1}{2}+i\tau,\frac{1}{2}-i\tau\,;\mu+1;-x)|\;dx+$$

$$E\int_{N_2}^{N}\frac{\tau^{2\tau-\frac{1}{2}}\,|S(\mu,\tau)|}{|P\,(-(\mu+\frac{1}{2})^2-\tau^2)\,Q\,(-(\mu+1/2)^2-\tau^2)|}\,d\tau.$$

Now, it is not difficult to prove the boundedness of the first and the second integrals above. For the third one, taking into account that , for $\tau \to \infty$,

$$|\,S(\mu,\tau)\,|\leq M\tau^{2\mu+1},\quad M>0,$$

this integral converges as $N \to \infty$.

Therefore,the integral (2.3) exists and thus, by the completeness of $\mathcal{D}'(\boldsymbol{I})$, there exists $f \in \mathcal{D}'(\boldsymbol{I})$ such that

$$\lim_{N\to\infty}\left\langle\int_0^N S(\mu,\tau)\,\mathbf{G}(\mu,\alpha,\tau,x)\,\frac{G(\tau)}{P\,(-(\mu+\frac{1}{2})^2-\tau^2)}\,d\tau,\,\phi(x)\right\rangle=\,<f,\phi>. \qquad \ldots (2.4)$$

The generalized function f determined in (2.4) is the restriction of $u \in \mathcal{E}'(\boldsymbol{I})$ to $\mathcal{D}'(\boldsymbol{I})$. In view of the continuity of the differentiation and the multiplication by x and by $\frac{1}{x}$ in $\mathcal{D}'(\boldsymbol{I})$, one can show that, for any $\phi \in \mathcal{D}(\mathrm{I})$

$$\lim_{N\to\infty}\left\langle P(A'_x)\int_0^N S(\mu,\tau)\,G(\mu,\alpha,\tau,x)\,\frac{G(\tau)}{P\,(-(\mu+\frac{1}{2})^2-\tau^2)}\,d\tau,\,\phi(x)\right\rangle=\,<P(A'_x)f,\phi>$$

and from this expression and the above inversion formula, it finally follows that

$$<g,\phi>\,=\langle\,P\,(A'_x)\,f,\,\phi\,\rangle$$

This result proves that the generalizd function $f \in \mathcal{D}'(\boldsymbol{I})$ is the restriction of $u \in \mathcal{E}'(\boldsymbol{I})$ to $\mathcal{D}(\boldsymbol{I})$ and that it satisfies the equation (2.1)

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REFERENCES

[1] A.Erdelyi, W. Magnus, F.Oberhettinger and F.G.Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.

[2] A. Erdelyi, W. Magnus,F. Oberhettinger, F.G. Tricomi,*Tables of Integral Transforms*, Vols.I and II, McGraw-Hill,New York,1954.

[3] N.Hayek, B.J.Gonzalez and E.R. Negrin, Una clase de transformada indice relacionada con la de Olevskii, Actas XIV Jornadas Hispano-Lusas de Matematicas,Univ. La Laguna Vol. I, (1989), 401-405.

[4] N. Hayek, B.J. Gonzalez and E.R. Negrin, Abelian theorems for the index ${}_2F_1$-transform, *Rev. Tecn. Fac. Ingr. Univ. Zulia* **15** (1992), 167-171.

[5] N. Hayek and B.J. Gonzalez, The index ${}_2F_1$-transform of generalized functions,Comment. Math. Univ.Carolinae, **34** (1993),657-671.

[6] B. van der Pol and H. Bremmer, *Operational Calculus*, Cambridge University Press, New York, 1964.

[7] A.H. Zemanian, *Generalized Integral Transformations*, Interscience Pubishers,New York,1968.

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