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ELECTROGRAVITATIONAL INSTABILITY OF A FLUID CYLINDER UNDER MODULATING VARYING ELECTRIC FIELD ON UTILIZING THE ENERGY PRINCIPLE

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ABSTRACT

The electrogravitational instability of a fluid circular jet dispersed in a self-gravitating tenuous medium of negligible motion pervaded by modulating general varying electric field has been developed on utilizing the Lagrangian energy principle. The fundamental equations describing the problem are deriving and solved in the unperturbed and perturbed states, the total changes in the electric, kinetic and self-gravitating energies are computed. The Lagrangian energy principle technique has been used and a result it is found that the system is governing by a second order integro-differential Mathieu euqation. Several categories have been analysed via this equation. The gravitational force is only destabilizing for small axisymmetric perturbation. The internal electric field penetrated the fluid cylnder has no direct influence on the stability of the fluid while the exterior longitudinal and transverse electric fields are being stabilizing or destabilizing according to restrictions. Resonance domains are appeared due to the field periodicity and in some regions the stability conditions depend only on the field frequency. The electric field frequency is stabilizing in a small axi-symmetric region and destabilizing otherwise. The amplitude of the modulating electric field could be fully stabilizing, under certain restrictions, and suppressing the destabilizing character of the other physical parameters and hence stability arises.

Numeruous reported works could be recovered with appropriate choices as limiting cases.

1. Introduction

The in stability of cylindrical fluid column endowed with surface tension or/and acted upon external forces such as electrodynamic or electromagnetic forces has been involved in several texts by a lot of researchers. Referring to these pioneering works see.

{[2],[3],[4],[6],[9],[13],[14],[15],[16],[21],}

The response of the axisymmetric instability of a self-gravitating cylinder ambient with self-gravitating vacuum was due to Chandrasekhar and Fermi [51]. It has a correlation with understanding the dynamical behaviour of the spiral arms of galaxies and sun spots. Chandrashekhar [4] studied such a problem in details for different cases in several literature. He [4] summarised his results along with those of others for different problems of different configuration models. Radwan [17] has extended that work [5] by studying the stability of a self- gravitating fluid cylinder dispersed in a self -gravitating fluid of different density. In the works [5] and [17] it is used totally different techniques where in the latter we have used the principle of energy.

The influence of the electrodynamic force on the self-gravitating fluid cylinder has been examined for first time by Radwan [18] . In this recent work [18] both the self-gravitating fluid and the surrounding vacuum are assumed to be pervaded by uniform (constant) electric fields. The aim of the present work is to investigate the electro-dynamic stability of a self-gravitating fluid cylinder such that the electric fields not only varying but also modulating with the same periodicity. This will be done for all axisymmetric and non-axisymmetric modes of perturbation, on using the Lagrangian energy principle technique.

2. Formulation of the Problem

Consider a dielectric self-gravitating fluid cylinder (of radius R_0) ambient with a tenuous dielectric medium of negligible motion. ε^i is the dielectric constant of the fluid matter and idem ε^e for the surrounding medium where from now on the superscripts i and e indicate interior and exterior the fluid jet. The fluid is assumed to be non-viscous, incompressible and of uniform mass density ρ. The fluid cylinder is being pervaded by the modulating electric field

$$\boldsymbol{E}_0^i = (0, 0, 1)E_0 \cos(w\,t) \qquad (1)$$

while the surrounding medium is assumed to be penetrated by the modulating general varying electric field.

$$\boldsymbol{E}_0^e = (0, \beta R_0 r^{-1}, \alpha)E_0 \cos(w\,t) \qquad \ldots (2)$$

where E_0 and w is the intensity and frequency of the electric field and α, β are parameters satisfying certain conditions. The components of $\boldsymbol{E}_0^i$ and $\boldsymbol{E}_0^e$ are considered along the utilizing cylindrical polar coordinates (r, ϕ, z) with the z-axis coinciding with the axis of the fluid cylinder. The fluid is acting up on the intertia, electrodynamic, pressure gradient and self- gravitating forces while the surrounding medium of the fluid is acting up on the self-gravitating and electrodynamic forces. We assume,initially, that there are no surface charges at the boundary surface and therefore the surface charge density will be zero during the perturbation. We also assume that the quasi- static approximation is valid for the problem under consideration.

The fundamental basic equation required for describing and analysing such kind of problems are coming out from the combination of the ordinary hydrodynamic equations together with those of Newtonian gravitational theory and with Maxwell's electrodynamics equations. For the problem under consideration, these basic equations may be formulated as follows.

In the fluid region :

The gravitational electrofluid dynamic vector equation of motion

$$\rho(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}.\nabla)\boldsymbol{u}) = (\varepsilon^{i}/2)\ \nabla(\boldsymbol{E}^{i}\ .\ \boldsymbol{E}^{i}) - \nabla_{p} - \rho\nabla\Phi^{i} \qquad \text{... (3)}$$

The equation of continuity expressing the conversation of mass

$$\nabla\ .\ \boldsymbol{u} = 0. \qquad \text{... (4)}$$

The self-gravitating Poisson's equation

$$\nabla^2\Phi^{i} = 4\pi G\rho. \qquad \text{... (5)}$$

The Maxwell's electrodynamic equations

$$\nabla\ .\ (\varepsilon^{i}\,\boldsymbol{E}^{i}) = 0, \quad \nabla\times\boldsymbol{E}^{i} = 0 \qquad \text{... (6), (7)}$$

In the surrounding tenuous medium :

The Laplace's equation for the gravitational potential

$$\nabla^2\,\Phi^{\varepsilon} = 0. \qquad \text{... (8)}$$

Maxwell's electrodynamic equations

$$\nabla\ .\ (\varepsilon^{e}\,\boldsymbol{E}^{e}) = 0, \quad \nabla\times\boldsymbol{E}^{e} = 0. \qquad \text{... (9), (10)}$$

Here p and $\boldsymbol{u}$ are the fluid kinetic pressure and velocity vector, G is the gravitational constant, Φ^{i} is the gravitational potential in the fluid region and idem Φ^{ε} in the surrounding medium and $\boldsymbol{E}$ is the electric field intensity.

3. Linearization and Solutions

In order to analyse such kind of study we assume that the fluid boundary surface is acting up on a sinusoidal deformation and consequently the location of the perturbed interface could be given in the form.

$$r = R_0 + \gamma(t)\cos(kz) + m\phi) \qquad \text{... (11a)}$$

Here $\gamma(t)$ is, some function of time t, the surface diaplacement, k (any real number) is the longitudinal wavenumber and m (an integer) is the azimuthal wavenumber. The second term in the right side of (11) is the elevation of the surface wave measured from the unperturbed position. Based on the perturbation form (11 a), for small departure from the equilibrium state every physical variable quantity $Q(r, \phi, z)$ could be expressed as

$$Q(r, c, z; t) = Q_0(r) + \gamma(t)\,Q_1(r, c, z) \qquad \text{... (11b)}$$

where Q_1 is the change in Q due to a perturbation.

It is intended to investigate the stability of this problem using an analytical perturbation technique on the basis of the Lagrangian energy principle. It is noted that the Lagrangian function L is constructed as

$$L = \Omega_1 - W_1 - V_1 \quad \text{... (12)}$$

and the Lagrangian second order differential equation is being

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\gamma}}\right) - \frac{\partial L}{\partial \gamma} = 0$$

where $\dot{\gamma}$ is the Lagrangian variable for the present problem and where the dot over γ means that γ is being differentiated with respect to time. The physical quantities Ω_1, W_1 and V_1 are the changes in Ω, W and V due to the perturbation of the boundary surface (11 a) with Ω is the total kinetic energy, W is the total electrical energy and V is the gravitational potential energy. One have to refer here that the quantities with subscript 1 mean quantities in the perturbation state while those with subscript o mean their value in equilibrium state.

In the perturbation state, the basic electrodynamic equations (6), (7), (9) and (10) degenerate to

$$\nabla \times \boldsymbol{E}_1^{i,e} = 0, \quad \nabla \cdot (\varepsilon \boldsymbol{E}_1^{i,e}) = 0 \quad \text{... (14a, b) (15a, b)}$$

From the viewpoint of the vector analysis, equations (14,a,b) mean that $\boldsymbol{E}_1^{i,e}$ could be derived from scalar electrical potentials.

$$\boldsymbol{E}_1^{i,e} = -\nabla \psi_1^{i,e} \quad \text{... (16a, b)}$$

Combining these vector equations together with equations (15 a, b), the electrical potentials $\psi_1^{i,e}$ satisfy the Laplace's equations.

$$\nabla^2 \psi_1^{i,e} = 0 \quad \text{... (17a, b)}$$

In cylindrical polar coordinate (r, ϕ, z) these equations may be rewritten as

$$r^{-1}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_1^{i,e}}{\partial r}\right) + r^{-1}\frac{\partial}{\partial \phi}\left(r^{-1}\frac{\partial \psi_1^{i,e}}{\partial \phi}\right) + r^{-1}\frac{\partial}{\partial z}\left(r\frac{\partial \psi_1^{i,e}}{\partial z}\right) = 0 \quad \text{... (18a, b)}$$

By revorting to the linear perturbation technique and based on the space time $(r, \phi, z; t)$ -dependence, the perturbed quantities $\psi_1^i, \psi_1^e, \psi_1^i\ \psi_1^e$ etc. can be expressed as $\cos(kz + m\phi)$ time an amplitude functon of r viz.,

$$\psi_1^{i,e}(r, c, z; t) = \gamma(t)\, \psi_1^{i,e}(r) \cos(kz + m\phi) \quad \text{... (19a, b)}$$

Inserting (19 a, b) into (18 a, b) we obtain

$$r^{-1}\frac{d}{dr}\left(r\frac{d\psi_1^{i,e}}{dr}\right)-(k^2+m^2r^{-2})\,\psi_1^{i,e}=0 \qquad \text{... (20a, b)}$$

The solutions of the second order ordinary differential equations (20 a,b) are given in terms of Bessel's functions with imaginary argument. Under the present circumstances, the non-singular for ψ_1^i as $r \to 0$ interior the fluid cylinder and for ψ_1^e as $r \to \infty$ exterior it are given by

$$\psi_1^i\,(r) = A^i\,(t)\,I_m(kr) \qquad \text{... (21)}$$

$$\psi_1^e\,(r) = A^e\,(t)\,K_m(kr). \qquad \text{... (22)}$$

where A^i and A^e are time dependent functions of integrations to be determined; $I_m(kr)$ and $K_m(kr)$ are the modified Bessel's functions of first and second kind of order m.

The basic gravitational potential equations (5) and (8) reduce to the following equations in the unperturbed state

$$\nabla^2\,\Phi_0^i = 4\pi G\rho \qquad \text{... (23)}$$

$$\nabla^2\,\Phi_0^e = 0. \qquad \text{... (24)}$$

These equations are solved on using the unperturbed simplifications of longitudinal and azimuthal symmetries $\partial/\partial z = 0$ and $\partial/\partial c = 0$. The solutions are matched across the boundry surface at $r = R_0$. Apart from the singular solutions the solution for Φ_0^i as $r \to 0$ interior the fluid cylinder and that one for Φ_0^e exterior the cylinder as $r \to \infty$ are given by

$$\Phi_0^i = \pi G\rho r^2 \qquad \text{... (25)}$$

$$\Phi_0^e = \pi\rho\,GR_0^2 + 2\pi G\rho\,R_0^2\log(r/R_0)^2 + C_0 \qquad \text{... (26)}$$

where C_0 is an additive constant with which we need not be further concerned.In the perturbed state, the basic equations (5) and (8) degenerate to the Laplace's equations

$$\nabla^2\,\Phi_1^{i,e} = 0. \qquad \text{... (27a, b)}$$

These equations could be solved on using similar steps as those which have already been used for solving equations (17 a, b). The non-singular solutions for Φ_1^i and Φ_1^e are given by

$$\Phi_1^i\,(r, \phi, z; t) = B^i\,\gamma(t)\;I_m(kr)\cos\,(kz + m\phi) \qquad \text{... (28)}$$

$$\Phi_1^e\,(r, \phi, z; t) = B^e\,\gamma(t)\,K_m\,(kr)\cos\,(kz + m\phi) \qquad \text{... (29)}$$

where B^i and B^e are constants of integration to be identified. It is worthwhile to mention here that we have originally considered departures from an unperturbed right cylindrical shape of an incompressible fluids. For this reason the argument of the sinusoidal acting wave $\sin(kz + m\phi + n\pi/2)$ where n is an integer appeared as $\cos(kz + m\phi)$ in the solutions (21),(22),(28)and (29).

Now, since the Lagrangian coordinate γ is a function of time t, each element of the fluid will execute a motion. Such a motion may be derived from the Lagrangian displacement

$$\boldsymbol{u} = \partial\xi/\partial t. \qquad \ldots (30)$$

However, taking the divergence of the perturbed equation of motion which resulting equation (3), linearizing and using the incompressibility condition ($\nabla \cdot \boldsymbol{u}_1 = 0$), we obtain

$$\nabla^2 S_1 = 0, \quad S_1 = p_1/\rho + \Phi^i - (\varepsilon^{i/2})\,(\boldsymbol{E}.\boldsymbol{E})^i_1. \qquad \ldots (31, 32)$$

The non-singular solution of (31) is given by

$$S_1(r, \phi, z; t) = C^i\,\gamma(t)\,I_m(kr)\cos(kz + m\phi) \qquad \ldots (33)$$

where C^i is an unspecified constant of integration.

4. Boundary Conditions

The solutions (19 a,b), (21),(22),(25) to (29) and (33) of the fundamental equations (3) to (10) must satisfy certain boundary conditions across the perturbed interface (11 a) at $r = R_0$. These appropriate boundary conditions could be stated as follows.

4.1. The electrodynamic conditions

(1) The electric potential ψ must be continuous across the perturbed interface (11 a) at $r = R_0$. This condition, on using (21) and (22), yields

$$A^e(t) = [I_m(x)/K_m(x)]\,A^i(t) \qquad \ldots (34)$$

where $x(=kR_0)$ is the dimensionless longitudinal wavenumber.

(2) The nomral component of the electric displacement must be also continuous across the perturbed boundary surface (11 a) at $r = R_0$ i.e.

$$\boldsymbol{N} \cdot (\varepsilon^i \boldsymbol{E}^i - \varepsilon^e \boldsymbol{E}^e) = 0. \qquad \ldots (35)$$

Here $\boldsymbol{N}$ is, the outward unit vector normal to the perturbed interface (11 a), given by

$$\phi = \nabla F(r, \phi, z; t) / |\nabla F(r, c, z; t)| \qquad \ldots (36)$$

where $F(r, c, z; t) = 0$ is the equation of the boundary surface given by

$$F(r, \phi, z; t) = r - R_0 - \gamma(t)\cos(kz + m\phi) \qquad \ldots (37)$$

Thus

$$N = N_0 + N_1$$

$$= (1, 0, 0) + (0, -mR_0, k)\ \gamma(t) \cos(kz + m\phi) \qquad \dots (38)$$

By the use of (1),(2),(11 a), (16 a,b), (19 a,b), (21), (22) and (38) for the condition (35), we obtain

$$\varepsilon^e K_m{}'(x) A^3 - \varepsilon^i I_m{}'(x) A^i = \left(\frac{ix}{k}(\varepsilon^i - \varepsilon^e) - \frac{im\beta}{k}\varepsilon^e\right) E_0 \cos wt \quad \dots (39)$$

Solving (34) and (39) for $A^i(t)$ and $A^e(t)$ we get

$$A^i(t) = \frac{[iX(\varepsilon^i = \varepsilon^e) - i\beta\varepsilon^e] K_m(x)}{k[\varepsilon^e I_m(x) K_m{}'(x) - \varepsilon^i I_m{}'(x) K_m(x)]} E_0 \cos(w\,t) \qquad \dots (40)$$

$$A^e(t) = \frac{ix(\varepsilon^i - \varepsilon^e) - i\beta\varepsilon^e]\, I_m(x)}{k[\varepsilon^e I_m(x) K_m'(x) - \varepsilon^i I_m{}'(x) K_m(x)]} E_0 \cos(w\,t). \qquad \dots (41)$$

4.2. Self-gravitating Conditons

(*i*) The gravitational potential Φ must be continuous across the perturbed fluid interface (11 a) at $r = R_0$. This condition gives

$$B^i I_m(x) = B^e K_m(x). \qquad \dots (42)$$

(*ii*) The derivative of the self-gravitating potential Φ must also be continuous across the surface (11 a) $r = R_0$. This condition, on using (11 a), (25),(26), (28) and (29) yields

$$B^i I_m{}'(x) - B^e K_m{}'(x) = 4\pi\rho\, G/k. \qquad \dots (43)$$

Solving (42) and (43) for B^i and B^ewe get

$$B^i = 4\pi\rho G R_0 K_m(x), \quad B^e = 4\pi\rho G R_0 I_m(x) \qquad \dots (44)$$

where use have been made of the Wronskian

$$W_m(I_m(x), K_m(x)) = I_m(x)\, K_m{}'(x) - I_m{}'(x)\, K_m(x) = -x^{-1} \qquad \dots (46)$$

in obtaining (44) and (45).

4.3. The Kinematic Condition

The normal component of the velocity of the fluid particles must be compatible with the velocity of the perturbed surface (11 a) at $r = R_0$ i.e.

$$u_{1r} = \frac{\partial r}{\partial t}\,. \qquad \dots (47)$$

By the use of the linearized form of the equation of motion (3)

$$\rho\,\frac{\partial \boldsymbol{u}_1}{\partial t} = -\nabla S_1 \qquad \dots (48)$$

and equations (30) to (33) we obtain

$$\xi_1 = -(C^i/\rho)\,\gamma(t)\,kI_m{}'(kr)\cos(kz+m\phi). \qquad \dots (49)$$

By an appeal to (11 a) and (47) to (49), the coefficient C^i is completely determined and the perturbed velocity vector of the fluid is being

$$\boldsymbol{u}_1 = [R_0^2/(x\,I_m{}'(x))]\,\frac{\partial\gamma(t)}{dt}\,\nabla\,[I_m(kr)\cos(kz+m\phi)] \qquad \dots (50)$$

5. Computation of Kinetic and Poetntial Energies

The change in the total kinetic energy Ω_1 (per unit length) of the fluid jet associated with the motion specified by (30) is given by

$$\Omega_1 = \tfrac{1}{2}\rho\int_0^{R_0}\int_{kz=0}^{2\pi}\int_Q^{2\pi}(\boldsymbol{u}.\boldsymbol{u})_1\,r\,dr\,\frac{dkz}{2\pi}\,dc \qquad \dots (51)$$

$$= (\pi\rho R_0^2/(2k^2))\,[I_m{}'(x)]^{-2}\left(\frac{d\gamma(t)}{dt}\right)^2 J_m(y) \qquad \dots (52)$$

where the integral $J_m(y)$ is defined by

$$J_m(y) = \int_0^x [(I_m{}'(y))^2 + (1+m^2y^{-2})\,(I_m(y))^2]\,y\,dy \qquad \dots (53)$$

This integration has been carried out on using the identity (which follows from Bessel's equations)

$$\frac{d}{dy}\,[y\,f_m(y)\,f_m{}'(y)] = y[f_m{}'(y))^2 + (1+m^2y^{-2})\,(f_m(y))^2] \qquad \dots (54)$$

where $f_m(y)$ stands for both the functions $I_m(x)$ and $K_m(x)$, and therefore

$$J_m(y)\;|_{r=R_0} \equiv x\,I_m(x)\,I_m{}'(x) \qquad \dots (55)$$

Consequently

$$\Omega_1 = \pi\rho\,R_0^4\left(\frac{I_m(x)}{2x\,I_m{}'(x)}\right)\left(\frac{d\gamma(t)}{dt}\right)^2. \qquad \dots (56)$$

Now suppose that the amplitude γ of the surface deformation (11 a) is increased by the increment $\delta\gamma$, then due to this infinitisimal increment in the amplitude of deformation the change δV in the gravitational potential energy can be determined by evaluating the work done during the displacement of the matter required to produce the change in γ. To evaluate this work it is necessary to specify quantitavely the redistribution which does take place. Arbitrary deformation of an incompressible fluid can be thought of as resulting from the Lagrangian displacement ζ_1 applied to each point of the fluid medium. We assume that the perturbed motion is irrotational and this is in fact because we have considered in the unperturbed state that the fluids are incompressible and non-viscous. Therefore, the Lagrangian displacement of the fluid could be expressed as

$$\zeta_1 = \nabla G_1. \qquad \dots (57)$$

Combining equation (57) together with the incompressible condition, we find that the displacement poetential G_1 satisfies Laplace's equation

$$\nabla^2 G_1 = 0. \qquad \dots (58)$$

The solution of this equation on using steps similar to those used for solving equations (17 a,b) is given by

$$G_1 = C_1 \gamma(t) I_m(kr) \cos(kz + m\phi). \qquad \dots (59)$$

The constant of integration C_1 could be determined by applying the condition states that the radial component of $\zeta_1$must reduce to $\gamma(t) \cos(kz + m\phi)$ at $r = R_0$. Therefore

$$\zeta_1 = \frac{R_0^2}{x I_m'(x)} \gamma(t) \nabla [I_m(kr) \cos(kz + mc)]. \qquad \dots (60)$$

Hence the corresponding displacement $\delta\zeta_1$ which must be applied to each point of the fluid in order to increase the amplitude of the deformation by $\delta\gamma$ is given by

$$\delta\zeta_1 = \frac{R_0^2 \, \delta\gamma(t)}{x I_m'(x)} \nabla [I_m(kr) \cos(kz + m\phi)]. \qquad \dots (61)$$

Now, due to that additional deformation $\delta\gamma$, the change in the total self-gravitating potential energy δV_1 (per unit length) can be identified by integrating the work done by the displacements $\delta\zeta_1$ in the gravitational potential Φ_1^i. This means that

$$\delta V_1 = 2\pi\rho << \int_0^{R_0 + \gamma(t)\cos(kz+mc)} (\delta\zeta_1 \cdot \nabla \Phi_1^i) \, r \, dr >> \qquad \dots (62)$$

where the angular brackets mean that the quantity enclosed should be averaged over c and z. Substituting from (28), (44) and (61) into (62) and carrying out the required integration we get

$$\delta V_1 = 2 \gamma(\delta\gamma) \rho G R_0^4 \pi^2 [\rho - \frac{2\rho K_m(x)}{x I_m'(x)} J_m(y)] \qquad \dots (63)$$

where $J_m(y)$ is given by (53) or rather by (55), Inserting (55) into (63) and integrating the resulting expression from zero to γ we get

$$V_1 = -2\pi^2 G \rho^2 R_0^4 (I_m(x) K_m(x) - \tfrac{1}{2}) \gamma^2. \qquad \dots (64)$$

which gives the required change in the total self-gravitating potential energy due to the deformation (11 a).

Now,our duty is to determine the change in the total electrical energy W_1 of the dielectric gravitational fluid cylinder dispersed in a dielectric tenuous medium of negligible motion.

We consider an electrostatic field $\boldsymbol{E}_0^e$ which has been established in the dielectic tenuous medium of dielectric constant ε^e. We assume that a dielectric body (a fluid cylinder) of a dielectric constant ε^i has been submerged into the field while the sources of $\boldsymbol{E}_0^e$ are maintained constant. Hence the electric energy of the dielectric body (fluid cylinder) in the electric field is given (by Stratton [20] as

$$W_0 = \frac{1}{2}(\varepsilon^e - \varepsilon^i)\int(\boldsymbol{E}_0^e.\,\boldsymbol{e}_0^i)\, r\, dr \qquad \ldots (65)$$

where $\boldsymbol{E}_0^i$ is the modified field distribution when the dielectric fluid column has been submerged in the medium. If the fluid cylinder is deformed and the electric field distribution now becomes, $\boldsymbol{E}_1^i$, then the electric energy of the deformed fluid cylinder is given by

$$W = \frac{1}{2}(\varepsilon^e \quad \varepsilon^i)\iiint(\boldsymbol{E}_0^e\,.\,\boldsymbol{E}_1^i)\, r\, dr\, d\phi\, dkz. \qquad \ldots (66)$$

Therefore, the change in the electric energy due to the performed deformation (11 a) is being

$$W_1 = W - W_0. \qquad \ldots (67)$$

Substituting from (1),(2),(16 a), (19 a), (21) and (40) into (65) and (66) and carrying out the required integrations we obtain

$$W_0 = (E_0^2\,\alpha^2/2)(\varepsilon^e - \varepsilon^i)\, R_0^2, \quad \text{at} \quad r = R_0$$

and

$$W = \frac{(\varepsilon^e - \varepsilon^i)}{2}\int_0^{2\pi}\int_{kz=0}^{2\pi}\int_0^{R_0}(\boldsymbol{E}_0^3\,.\,\boldsymbol{E}_1^i)\, r\, dr\, dkz\, d\phi. \qquad (69)$$

Carrying out the integrations in (69) and substituting from (68) and (69) into (67), the total change in the electric energy (per unit length) due to the deformation (11 a) of the fluid cylinder is given by

$$W_1 = \frac{\pi R_0^3 E_0^2 \cos^2 wt}{2}\left\{-\frac{[x\,\varepsilon^i - \varepsilon^e(x + m\beta)]^2\, I_m(x)\, K_m(x)}{x(\varepsilon^i\, I_m{}'(x)K_m(x) - \varepsilon^e K_m{}'(x)I_m(x)} + \varepsilon^e\,\beta^2\right\} \qquad \ldots (70)$$

6. Characteristic Value Problem

By the use of (56),(64) and (70) for (12), the Lagrangian function is construced and moreover if we use the Lagrangian equation (13) we obtain

$$\frac{d^2\gamma}{dt^2} - 4\pi G\rho \left(\frac{x\, I_m'(x)}{I_m(x)}\right)(I_m(x)\, K_m(x) - \tfrac{1}{2})\gamma$$

$$+ \frac{E_0^2\gamma}{\rho R_0^2} \cos^2(w\, t)\, \frac{z\, I_m'(x)}{I_m(x)} \left[\frac{[x(\alpha\varepsilon^e - \varepsilon^i) + m\beta\varepsilon^e]^2 I_m K_m}{[\varepsilon^i I_m'(x) K_m(x) - \varepsilon^e I_m K_m']} - \varepsilon^e\beta^2\right] = 0 \quad \text{... (71)}$$

Equation (71) is an integro-differential equation governing the surface displacement γ of the perturbation and may be rewritten in the following brief form

$$\frac{d^2\gamma}{d\eta^2} + (b - h^2 \cos^2\eta)\gamma = 0 \quad \text{... (72)}$$

where η and b are defined by

$$\eta = wt,\ b = \frac{4\pi G\rho}{2} \frac{x\, I_m'(x)}{I_m(x)} \left[\frac{1}{2} - I_m(x) K_m(x)\right] \quad \text{... (73, 74]}$$

$$h^2 = \frac{-E_0^2}{\rho R_0^2\, w^2} \left(\frac{x\, I_m'(x)}{I_m(x)}\right) \left\{\frac{(x\,\varepsilon^i - \varepsilon^e(\alpha x + m\beta))^2\, I_m(x) K_m(x)}{x[\varepsilon^i I_m(x) K_m(x) - \varepsilon^e I_m(x) K_m'(x)]} - \varepsilon^e\beta^2\right\} \quad \text{... (75)}$$

The second order ordinary differential equation (72) has the canonical form

$$\frac{d^2\gamma}{d\eta^2} + (a - 2q \cos 2\eta)\, \gamma = 0 \quad \text{... (76)}$$

where

$$a = b - \frac{1}{2}h^2, \quad q = \frac{1}{4}h^2. \quad \text{... (77)}$$

7. Limiting Cases

Under appropriate choices we can obtain the following limiting cases :

If we put $\beta = 0$ and $G = 0$ in equation (71) we recover the dispersion relation of Reynolds [19] (eqn (44)) if we neglect the surface tension and charges there and assume that fluid cylinder is ambient with vacuum medium.

If we suppose that $G = 0, \beta = 0$ and $w = 0$, equation (71) degenerates to that of N { symbol f }ayyar and Murty [12] (eqn (23)) if we neglect the capillary force there.

Inserting $\beta = 0$ and $G = 0$ in equation (71) we recover the characteristic equation of Abouelmagd and Nayyar [1] (eqn (2.7)) if we neglect the surface tension influence there.

Following Chandrashekhar and Fermi [5] we postulate that $\gamma \sim \exp\,(\sigma\, t)$ where σ is the growth rate, and moreover we assume here that $E_0 = 0$, $\beta = 0$, $\alpha = 0$, $w = 0$ and simultaneously $m = 0$; equation (72) yields

$$\sigma^2 = 4\pi\rho\, G \frac{x\, I_1(x)}{I_0(x)} \left((I_0(x)\, K_0) - \tfrac{1}{2}\right), \qquad \dots (78)$$

where use has been made of the relation $I_0'(x) = I_1(x)$. The dispersion relation (78) has been derived for first time by Chandrashekhar and Fermi [5] for the aim of investigating the self-gravitating fluid cylinder dispersed into a self-gravitaing vacuum. Indeed they [5] have utilized a totally different technique from that used here. Their perturbation technique [5] is mainly depends on representing the solenoidal vector fields in terms of poloidal and toroidal vector fields, and that analysis is only valid for axisymmetric perturbation mode $m = 0$ but not for those of non- axisymmetric $m \neq 0$.

If we postulate $\gamma \sim \exp(\sigma t)$ and we moreover assume that $E_0 = 0$, $m \neq 0$, $\beta = 0$, $\alpha = 0$ and $w = 0$, the characteristic equation (72) yields

$$\sigma^2 = 4\pi G\rho \frac{x I_m'(x)}{I_m(x)} \left(I_m(x)\, K_m(x) - \tfrac{1}{2}\right). \qquad \dots (79)$$

This dispersion relation is derived and discussed by Chandrasekhar [4] on utilizing the noraml mode analysis, and did mention the correlation of such study with understanding the dynamical behaviour of the spiral arms of galaxies.

If we postulate that $\gamma \simeq \exp(\sigma t)$ and we put $\beta = 0$ and $w = 0$, we obtain from (72) an electrogravitational dispersion relation of a fluid cylinder pervaded by uniform electric field $(0, 0, E_0)$ dispersed in gravitating medium penetrated by the longitudinal uniform electric field $(0, 0, \alpha E_0)$, see Radwan [18].

If we assume that $\gamma \simeq \exp(\sigma t)$ and simultaneously we suppose that $w = 0$, $\alpha = 0$ and $m \geq 0$, the characteristic equation (72) degenerates to a dispersion relation of a self-gravitating fluid cylinder pervaded by $(0, 0, E_0)$ and surrounded by the varying electric field $(0, \beta R_0 r^{-1} E_0, 0)$.

Other stability criteria could be derived in the cases :

(*i*) As $G = 0$, $E_0 \neq 0$, $\beta \neq 0$, $\alpha = 0$, $w = 0$ and $m = 0$.

(*ii*) As $G = 0$, $E_0 \neq 0$, $\beta = 0$, $\alpha \neq 0$, $w = 0$ and $m = 0$.

(*iii*) The cases (*i*) and (*ii*) as $m \geq 0$.

8. Stability Discussions

The integro-differential equation (72) or rather its canonical form (76) is the Mathieu second order differential equation and its solution is given in terms of Mathieu functions. These transcendental functios (their behaviour, numerical data, ... etc.) are studied in several text books, see for example Morse and Feshback [11] and Mclachlan [10]. We have to mention here that our model of the gas-core fluid cylinder is being stable if the solution of equation (72) is periodic and this could be occurred under certain restrictions. These appropriate restrictions are depending on the relationship which correlate the parameters q and α. In numerical studies presenting q on a horizontal axis and α as vertical axis, it is found that the (q, α)-plane is classified into different categories. These categories are corresponding to stable and unstable regions bounded the characteristic curves of the Mathieu functions, see Ince [8] or/and Mclachlan [10]. Therefore we predict that the model of the gas-core fluid jet is stable if the solution of the Mathieu differential equation (72) is obtained such that the point (q, α) lies interior or/and on the boundaries of a stable domain in the stable regions and vice versa. Hence the condition for stability degenerates to the problem of the boundary regions of Mathieu function. Mclachlan /10/ gave the explicit condition

$$\left| D(0) \sin^2 \frac{\pi u}{2} \right|^{1/2} < 1 \qquad \text{... (79)}$$

for stability where $D(0)$ is the Hill's determinant. However the analysis of this condition is useless and rather cumbersome because $D(0)$ is infinite.

The numerical discussions and investigations concerning the stable and unstable regions of the characteristic curves of the Mathieu functions reveal, for very small values of q, that the first unstable region is bounded by the curves

$$\alpha = 1 \pm q \qquad \text{... (80)}$$

while the boundary curves of the other unstable regions which are higher than the first unstable regions are totally different from (80). However, in this respect, Morse and Feshbach [11] have found out an approximate relation for identifying the (in-) stability states. That aproximate relation is, valid only for very small values of h^2 being

$$(h^4 - 16(1-b)h^2 + 32b(1-b)) \geq 0 \qquad \text{... (81)}$$

If the restrictions (81) are satisfied the model must be stable and vice versa where the equality is corresponding to the marginal (neutral) stability. One have to refer here that $|h^2|$ is very small as the periodic field frequency w is very high. The inequality (81) is quadratic in h^2 and could be expressed as

$$(h^2 - \alpha_1)(h^2 - \alpha_2) \geq 0. \qquad \text{... (82)}$$

with

$$\alpha_1 = 8(1-b) - D \text{ and } \alpha_2 = 8(1-b) + D \qquad \text{... (83), (84)}$$

are the two roots of the equality in the restrictions (81) where

$$D = \{32\,(1-b)(2-3b)\}^{1/2}. \qquad \text{... (85)}$$

It is found more convenient to study and write down some properties of the modified Bessel functions before carrying out the instability and oscillation investigations of the present problem.

Consider the recurrence relation (cf. Gradshteyn and Ryzhik [7], of the modified Bessel functions

$$2\,I_m'(x) = I_{m+1}(x) + I_{m-1}(x) \qquad \text{... (86)}$$

$$2\,K_m'(x) = -\,K_{m+1}(x) - K_{m-1}(x). \qquad \text{... (87)}$$

Also consider the facts, for every non-zero real value of x, that $I_m(x)$ is positively definite and monotonic increasing while $K_m(x)$ is monotonic decreasing but never negative i.e.

$$I_m(x) > 0, \quad K_m(x) > 0 \qquad \text{... (88, 89)}$$

By the use of the relations (86) and (87) and the inequalities (88) and (89) we can show, for every $x \neq 0$, that

$$I_m'(x) > 0, \quad K_m'(x) < 0. \qquad \text{... (90, 91)}$$

Consequently for any positive values of ε^i and ε^e and $x \neq 0$, we get

$$I_m(x)\,K_m(x) > 0 \qquad \text{... (92)}$$

$$x(I_m'(x)/I_m(x)) > 0 \qquad \text{... (93)}$$

$$[\varepsilon^i\,I_m'(x)\,K_m(x) - \varepsilon^e\,I_m(x)\,K_m'(x)] > 0. \qquad \text{... (94)}$$

However, it is found numerically concerning the inequality (92), (see Chandrasekhar [4]), that

$$I_m(x)\,K_m(x) < \tfrac{1}{2} \text{ as } m \geq 1. \qquad \text{... (95)}$$

In the axisymmetric mode of peryurbation $m = 0$, the value of the compound functions $I_0(x)\,K_0(x)$ may be larger or smaller than $\frac{1}{2}$ and that depends on the x value. Moreover it is found that

$$I_0(x)\,K_(x) \leq \tfrac{1}{2} \text{ as } \quad 1.0668 \leq x < \infty \qquad \text{... (96)}$$

$$I_0(x)\,K_0(x) \geq \tfrac{1}{2} \text{ as } \quad 0 < x < 11.0668 \qquad \text{... (97)}$$

In view of the foregoing relations and inequalities, using (75), we can show that

$$h^2 < 0$$

provided that

$$\{x[\varepsilon^i I_m'(x) K_m(x) - \varepsilon^e I_m(x) K_m'(x)]\,\varepsilon^e\,\beta^2\} < ([x\varepsilon^i - \varepsilon^e(\alpha x + m\,\beta)]^2\, I_m(x) K_m(x))$$

Now, by an appeal to the inequalities (92), (93) and (95) to (98) for discussing the dispersion relation (79), we may identify the self-gravitating force influence. Following Chandrasekhar's analysis [4], it is found that the model is purely self-gravitating stable in the modes $m \geq 1$ for all $x \neq 0$ and also in the mode $m = 0$ in the domains $1.0668 \leq x < \infty$. It is only self-gravitating unstable in the axisymmetric mode $m = 0$ in the domain $0 < x < 1.0668$.

In order to determine the influences of the different acting electrodynamic forces on the present model with neglecting the periodicity of the basic electric fields, we would have to use the stability criterion

$$\frac{\rho R_0^2 \sigma^2}{E_0^2 \varepsilon^e} = \frac{x I_m'(x)}{I_m(x)} \left[\beta^2 - \frac{[x(\varepsilon^i/\varepsilon^e) - (\alpha x + m\beta)]^2 I_m(x) K_m(x)}{x[(\varepsilon^i/\varepsilon^e) I_m'(x) K_m(x) - I_m(x) K_m'(x)]} \right] \quad \text{... (99)}$$

which is coming out from (71) by considering $\gamma \sim \exp(\sigma t)$ and $G = 0$. The quantity $R_0(\rho)^{1/2}/(E_0(\varepsilon^e)^{1/2})$ has a unit of time therefore, the relation (99) is a dimensionless equaiton. In equation (99) there is no any term free from the parameters α and β. This means that the interior electric field $\boldsymbol{E}_0^i$ has no influence on the stabilty of the present problem. The influence of the axial exterior electric field $(\boldsymbol{E}_0^e)_z$ is represented by the term, including α in equation (99) as $\beta = 0$,

$$\frac{x^2[(\varepsilon^i/\varepsilon^e) - \alpha]^2 I_m'(x) K_m(x)}{I_m(x) K_m'(x) - (\varepsilon^i/\varepsilon^e) I_m'(x) K_m(x)]}. \quad \text{... (100)}$$

By the use of the inequalities (89), (90), and (94), we find that the axial exterior electric field is stablizing

$$\varepsilon^i > 0, \quad \varepsilon^e > 0, \quad \alpha > 0. \quad \text{... (101)}$$

This result is valied for all values of $x \neq 0$ in the axisymmetric mode $m = 0$ and also in those of non-axisymmetric $m \neq 0$.

The effect of the azimuthal exterior electric field $(\boldsymbol{E}_0^e)_\phi$ is represented by the terms, including β in equation (99) as $\alpha = 0$,

$$\frac{x I_m'(x)}{I_m(x)} \beta^2 \left[1 - \frac{m^2 I_m(x) K_m(x)}{x[(\varepsilon^i/\varepsilon^e) I_m'(x) K_m(x) - I_m(x) K_m'(x)]} \right] \quad \text{... (102)}$$

In the axisymmetric mode $m = 0$, it is purely destabilizing due to its contribution $\beta^2(x I_1(x)/I_0(x))$ where we have used the relation $I_0'(x) = I_1(x)$.

In the non-axisymmetric perturbation modes $m \geq 1$, the exterior azimuthal electric field has a strong stabilizing influence if, for each non-zero real value of x, the restrictions

$$(m^2 I_m(x)K_m(x)) \leq x[(\varepsilon^i/\varepsilon^e)I_m'(x)K_m(x) - I_m(x)K_m'(x)] \quad \dots (103)$$

are satisfied and vice versa where use has been made of the inequalities (88) to (94).

We may conclude here that the electrodynamic forces (with general variable electric field) acting on the present model are stabilizing if

$$x(\varepsilon^i I_m'(x)K_m(x) - \varepsilon^e I_m(x)K_m'(x))\beta^2 \leq [x\varepsilon^i - \varepsilon^e(\alpha x + m\beta)]^2 I_m(x)K_m(x) \quad \dots. (104)$$

where the equalities are associated witht he neutral stabilities. Indeed these general restrictions are deduced without paid any atention for the magnitude of E_0. The effect of the latter will be determined later through the investigation of the integro- differential equation (72) or rather its canonical form (76) and this will be our present scope.

Now, let us return to the general characteristic Mathieu differential equation (72). Following Morse and Feshbach [20] principles with taking into accounts the relations (83) to (85) for the inequalities (81) and (82), the (in-) stability investigations and discussions should be carried out in the following different cases.

The case $0 < b < (2/3)$

In this case we have $b > 0$ and simultaneously $3b < 2$ we can show that D is real. Hence we can prove that α_1 is positive, note that if α_1 is positive, then α_2 is also positive because $\alpha_2 > \alpha_1$ as follows. If α_1 is negative, we have from (83) that $(8(1-b) - D) < 0$ and on using (85) we get $b > 1$ which is a contradiction to the present postulates that $0 < b < (2/3)$ of the our case. Therefore, α_1 must be positive and α_2 is so. Now since α_1 and α_2 are real and positive we predict, on using (98), that each of $(h^2 - \alpha_1)$ and $(h^2 - \alpha_2)$ is negative, Hence the product $(h^2 - \alpha_1)(h^2 - \alpha_2)$ is positive definite and this shows that the restrictions (81) are statisfied under the conditions $0 < b$ and $3b < 2$.

The case $(2/3) < b < 1$.

This case is restricted by the two conditions $(2/3) < b$ and $b < 1$ simultaneously. Using these together with (98) we can prove that the two roots α_1 and α_2 are complex. Consequently the stability restrictions (81) are indentically satisfied.

Summarizing the results of the foregoing cases $0 < b < (2/3)$ and $(2/3) < b < 1$, we see that the stability restrictions (81) are satisfied and hence the model at hand is stable if b is bounded by

$$0 < b < 1. \quad \dots (105)$$

Combining (74) and (105), we conclude that the modulating dielectric self-gravitating fluid cylinder is stable if the frequency of the periodic electric field is larger than a critical value w_c such that

$$w_c^2 = \left[4\pi G\rho \frac{xI_m'(x)}{I_m(x)} (\tfrac{1}{2} - I_m(x)\, K_m(x)) \right] > 0 \qquad \text{... (106a)}$$

This restriction is independent of the amplitude of the applied electric field. Therefore, upon choosing appropriate value of W the model could be in stabilizing state. Hence an investigation of the right side in the equality (106a) is leading to find out exactly where are the domains of stability and those of instability. This is our present scope.

The equality in (106a) may be rewritten in the non-dimension form

$$\frac{w^2}{4\pi G\rho} = \frac{xI_m'(x)}{I_m(x)} (\tfrac{1}{2} - I_m(x)\, K_m(x)). \qquad \text{... (106b)}$$

By the use of the inequalities (93) and (95) for this relation, we conclude that the electirc field frequency is strongly destabilizing in all non-axisymmetric perturbation modes $m \geq 1$. Also in using the inequality (93) in the axisymmetic mode $m = 0$ of perturbation and taking into account the inequalities (96) and (97) for disscusing the relation (106 b) we deduce the following. The electric field frequency w has strong stabilizing influence in the doman $0 < x \leq 1.0668$ 1.0668 while it has a destabilizing effect in the doman $1.0668 < x < \infty$ upon axisymmetirc perturbation.

Therfore, we conclude that the frequency of the electric field is stabilizing for axisymmetric mode in the doman $0 < x \leq 1.0668$ only and destablizing otherwise.

The case $b > 1$.

In this case we have $b > 1$ so $3b > 2$ and we can also show that the determinant D is being real. Moreover we can show that α_1 is negative here as follows. If α_1 is positive, using (83), we get

$$8(1 - b) \geq D. \qquad \text{... (107)}$$

From the view point of (85), the restrictions (107) may be rewritten as

$$\left(64(1 - b)^2\right) \geq \left(32(1 - b)\,(2 - 3b)\right) \qquad \text{... (108)}$$

from which we obtain $b < 1$. The latter result is a contradiction to the supposition of the present case that $b > 1$, hence α_1 must be negative. In similar steps we can show that α_2 is positive, hence on using the inequality (98) we have

$$\left(h^2 - \alpha_2\right) < 0 \qquad \text{... (109)}$$

for all values of b (> 1). By the use or (109), the stability restricitions (82) would be satisfied only if

$$h^2 \geq \alpha_1 \qquad \ldots (110)$$

or alternatively

$$E_0^2 \geq \frac{\rho^2 R_0^2 w^2}{F_m^2(x)} [8(1-b)+D] \qquad \ldots (111)$$

with

$$F_m^2(x) = \frac{x I_m'(x)}{I_m(x)} \left\{ \frac{x\varepsilon^i - \varepsilon^e(\alpha x + m\beta)]^2 I_m(x) K_m(x)}{x[\varepsilon^i I_m'(x) K_m(x) - \varepsilon^e I_m(x) K_m'(x)]} - \varepsilon^e \beta^2 \right\} \qquad \ldots (112)$$

where use has been made of (75) and (83).

The case $b < 0$

This case in which b is assumed to be negative looks like the previous case in which $b > 1$ where we can show that

$$\alpha_1 = -\text{ve} \text{ and } \alpha_2 - +\text{ve} \qquad \ldots (113)\ (114)$$

Consequently the stability restrictions (82) are satisfied under the validity of the inequalities (109) and (110).

Therefore, we conclude that for the cases $b < 0$ and $b > 1$, the dielectric fluid cylinder will be stable if there exist a critical value E_0^C of the electirc field intensity E_0 such that $E_0 \geq E_0^C$ where

$$E_c = \frac{w R_0 \rho}{F_m(x)} (8(1-b)+D)^{1/2}. \qquad \ldots (115)$$

It is worthwhile to mention here that the quantity $(\varepsilon^e E_0^2/(\rho R_0^2))^{-1/2}$ has a unit of "time" and therefore we may formulate the characteristic relation (115) in a dimensionless form. Thence by giving appropriate values for the occurred different parameters in regular steps of the wavenumber x, we could find out the critical value of E_p i.e. E_0^c above which the instability character of the model is completely suppressed and stability then arises and sets in.

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