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MEROMORPHIC MULTIVALENT FUNCTIONS WITH FIXED COEFFICIENTS

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ABSTRACT

In the present paper we have obtained coefficient inequality for the class $Q_k^*(p, A, B)$ which consists of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \; (a_{p+n} \geq 0)$$

which are regular and p-valent in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. Further we have shown that the class $Q_k^*(p, A, B)$ is closed under arithmetic mean and convex linear combinations. Lastly we have obtained the radius of convexity for the class $Q_k^*(p, A, B)$. Various results obtained in the present paper are shown to be sharp.

1. Introduction

Let $Q(p)$ denotes the class of functions of the type

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n} \quad \text{.... (1.1)}$$

which are regular and p-valent in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. Further, let $Q^*(p)$ denotes the subclass of $Q(p)$ consisting of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n} \text{ and satisfying } a_{p+n} \geq 0. \quad \text{... (1.2)}$$

In [2] Uralegaddi-Ganigi introduced a class $Q^*(p, A, B)$ consisting of functions of type (1.2) and satisfying the condition

$$\left| \frac{zf'(z)/f(z) + p}{Bzf'(z)/f(z) + Ap} \right| < 1, \; |z| < 1 \quad \text{... (1.3)}$$

where

$$-1 \leq A < B \leq 1, \; A + B \geq 0.$$

Meromorphic univalent functions have been rather extensively studied by several authors like Uralegaddi [3] and Uralegaddi- Ganigi [2]. We begin by recalling the following lemma due to Uralegaddi-Ganigi [2].

Lemma 1 Let the function $f(z)$ be defined by (1.2). Then the function $f(z)$ is in the class $Q_k^*(p, A, B)$ if and only if

$$\sum_{n=0}^{\infty} \{(1+B)n + (2+A+B)p\}\, a_{p+n} \leq p(B-A). \qquad \text{... (1.4)}$$

In view of Lemma 1, we observe that $f(z)$ given by (1.2) in the class $Q_k^*(p, A, B)$ satisfy

$$a_p \leq \frac{(B-A)}{(2+A+B)}. \qquad \text{... (1.5)}$$

Hence we may take

$$a_p = \frac{(B-A)k}{(2+A+B)}, \; 0 \leq k \leq 1. \qquad \text{... (1.6)}$$

Let $Q_k^*(p, A, B)$ denotes the subclass of $Q_k^*(p, A, B)$ consisting of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad \text{.... (1.7)}$$

where

$$a_{p+n} \geq 0 \text{ and } 0 \leq k \leq 1.$$

In the present paper we have obtained coefficient inequality for the class $Q_k^*(p, A, B)$. Further we show that the class $Q_k^*(p, A, B)$ is closed under arithmetic mean and convex linear combinations. Lastly we have obtained radius of convexity for the class $Q_k^*(p, A, B)$. Various results obtained in this paper are shown to be sharp. Techniques used are similar to those of Silverman and Silvia [1].

Main Results

2. Coefficient Inequality

Theorem 1 Let the function $f(z)$ be defined by (1.7). Then is in the class $Q_k^*(p, A, B)$ if and only if

$$\sum_{n=1}^{\infty} \{(1+B)n + (2+A+B)p\} a_{p+n} \leq p(B-A)(1-k). \qquad \text{... (2.1)}$$

The result is sharp.

Proof Putting

$$a_p = \frac{(B-A)\,k}{2+A+B},\quad 0<k\le 1 \qquad \ldots (2.2)$$

in (1.4) and simplifying we get result. The result is sharp for the function

$$f(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \frac{(B-A)\,p(1-k)}{\{(1+B)n+(2+A+B)p\}} z^{p+n},\quad n\ge 1. \qquad \ldots (2.3)$$

Corollary Let the function $f(z)$ given by (1.7) be in the class $Q_k^*(p, A, B)$. Then

$$a_{p+n} \le \frac{(B-A)\,p(1-k)}{\{(1+B)n+(2+A+B)p\}},\ n\ge 1. \qquad \ldots (2.4)$$

3. Closure Theorem

In this section we shall show that the class $Q_k^*(p, A, B)$ is closed under arithmetic mean and convex linear combinations.

Theorem 2 Let

$$f_j(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \sum_{n=1}^{\infty} a_{p+n,j}\, z^{p+n} (a_{p+n,j} \ge 0) \qquad \ldots (2.5)$$

be in the class $Q_k^*(p, A, B)$ for every $j = 1, 2, \ldots, m$, then the function

$$g(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \sum_{n=1}^{\infty} b_{p+n}\, z^{p+n}\quad (b_{p+n} \ge 0) \qquad \ldots (2.6)$$

is also a member of $Q_k^*(p, A, B)$, where

$$b_{p+n} = \frac{1}{m} \sum_{j=1}^{m} a_{p+n,j}.$$

Proof Since $f_j(z) \in Q_k^*(p, A, B)$ it follows from Theorem 1 that

$$\sum_{n=1}^{\infty} \{(1+B)n + (2+A+B)p\}\, a_{p+n,j} \le (b-A)p\,(1-k) \qquad \ldots (2.7)$$

for every $j = 0, 1, 2, \ldots, m$. Hence

$$\sum_{n=1}^{\infty} \{1+B)n + (2+A+B)p\}\, b_{p+n}$$

$$= \sum_{n=1}^{\infty} \{(1+B)n + (2+A+B)p\} \left[1/m \sum_{j=0}^{m} a_{p+n,j} \right]$$

$$= \frac{1}{m} \sum_{j=0}^{m} \sum_{n=1}^{\infty} \{(1+B)n + (2+A+B)p\}\, a_{p+n,j}$$

$$\le (B-A)p\,(1-k)$$

$$\leq (B-A)\,p(1-k)$$

and the result follows.

Theorem 3 Let

$$f(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p$$

and

$$f_{p+n}(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \frac{(B-A)\,p(1-k)}{\{(2+A+B)p+(1+B)n\}} z^{p+n}, \ (n \geq 1).$$

Then $f(z) \in Q^*(p, A, B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=-1}^{\infty} \lambda_{p+n} f_{p+n}(z).$$

where

$$\lambda_{p+n} \geq 0 \ \text{ and } \ \sum_{n=-1}^{\infty} \lambda_{p+n} = 1.$$

Proof Let

$$f(z) = \sum_{n=-1}^{\infty} \lambda_{p+n} f_{p+n}(z),$$

$$f(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \sum_{n=1}^{\infty} \frac{(B-A)\,p(1-k)}{\{(1+B)n+(2+A+B)p\}} z^{p+n}.$$

Since

$$\sum_{n=1}^{\infty} \frac{(B-A)\,p(1-k)\ \lambda_{p+n}}{\{(1+B)n+(2+A+B)p\}} \frac{\{(1+B)n+(2+A+B)p\}}{(B-A)\,p(1-k)}$$

$$= \sum_{n=1}^{\infty} \lambda_{p+n} = 1 - \lambda_p \leq 1.$$

Hence, by Theorem 1, $f(z) \in Q_k^*(p, A, B)$.

Conversely, suppose that

$$f(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \ (a_{p+n} \geq 0)$$

is in the class $Q_k^*(p, A, B)$. Then by using (2.4) we get

$$a_{p+n} \leq \frac{(B-A)\,p(1-k)}{\{(1+B)n+(2+A+B)p\}}, \ n \geq 1.$$

Setting

$$\lambda_{p+n} = \frac{\{(1+B)n+(2+A+B)p\}}{(B-A)\,p(1-k)} a_{p+n} \ (n \geq 1).$$

and

$$\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_{p+n}$$

we have

$$f(z) = \sum_{n=-1}^{\infty} \lambda_{p+n} f_{p+n}(z).$$

4. Radius of Convexity

Theorem 4

Let $f(z)$ defined by (1.7) be in class $Q^*k(p, A, B)$. Then $f(z)$ is mermorphically p-valent convex in $0 < |z| < r = r(p, A, B, K)$, is the largest value for which

$$\frac{3p^2 (B-A) kr^{2P}}{(2+A+B)} + \frac{(p+n)(3p+n)(B-A)p(1-k)}{\{(1+B)n + (2+A+B)p\}} r^{2p+n} \le p^2,$$

$(n = 1, 2, ...).$

The result is sharp for the function

$$f_{p+n}(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \frac{(B-A)p(1-k)}{\{(1+B)n + (2+A+B)p\}} z^{p+n},$$

for some n.

Proof It is sufficient to show that

$$\left| \frac{\{zf'(z)\}' + pf'(z)}{f'(z)} \right| \le p \text{ for } 0 < |z| \; r(p, A, B, k).$$

Note that

$$\left| \frac{[zf'(z)]' + pf'(z)}{f'(z)} \right|$$

$$\le \frac{\frac{2p^2(B-A)k}{(2+A+B)} r^{2p} + \sum_{n=1}^{\infty} (p+n)(2p+n) a_{p+n} r^{2p+n}}{p - \frac{(B-A)kp}{(2+A+B)} r^{2p} - \sum_{n=1}^{\infty} (p+n) a_{p+n} r^{2p+n}} \le p$$

for $0 < |z| \le r$ if

$$3p^2 \frac{(B-A)k}{2+A+B)} r^{2p} + \sum_{n=1}^{\infty} (p+n)(3p+n) a_{p+n} r^{2p+n} \le p^2.$$

Since $f(z) \in Q^*k(p, A, B)$, we may take

$$a_{p+n} = \frac{(B-A)p(1-k)\lambda_{p+n}}{\{(1+B)n + (2+A+B)p\}}, \quad \sum_{n=1}^{\infty} \lambda_{p+n} \le 1.$$

For each fixed r, choose an integer $n = n(r)$ for which

$\frac{(p+n)(3p+n)}{(2+A+B)} r^{2p+n}$ is maximal. Then,

$$\sum_{n=1}^{\infty} (p+n)(3p+n)\, a_{p+n}\, r^{2p+n} \leq \frac{(p+n)(3p+n)(B-A)(1-k)\, r^{2p+n}}{\{(1+B)n + (2+A+B)p\}}.$$

Now find the value $r_0 = r_0\,(p, A, B, k)$ and corresponding $n(r_0)$ so that

$$\frac{3p^2(B-A)k}{(2+A+B)} r_0^{2p} + \frac{(p+n)(3p+n)(B-A)(p(1-k)}{\{(1+B)n + (2+A+B)p\}} r_0^{2p+n} = p^2.$$

This is the value of r_0 for which $f(z)$ is convex in $0 < |z| < r_0 = r_0\,(p, A, B, k)$.

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