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RAYLEIGH-TAYLOR INSTABILITY OF ROTATING OLDROYDIAN VISCOELASTIC FLUIDS IN POROUS MEDIUM IN PRESENCE OF A VARIABLE MAGNETIC FIELD

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ABSTRACT

The Rayleigh-Taylor instability of Oldroydian viscoelastic fluid in porous medium in the presence of uniform rotation and variable magnetic field is considered. The magnetic field, the viscosity and the density are assumed to be exponentially varying. For stable density stratification, the system is found to be stable for distrubances of all wave numbers. The magnetic field stablizes the potentially unstable stratification for small wave- length perturbations which are otherwise unstable. The long wave length perturbations remain unstable and are not stablized by magnetic field. Rotation does not affect the stability or instability, as such, of a stratification.

1. Introduction

A detailed account of the instability of the plane interface between two Newtonian fluids, under varying assumptions of hydrodynamics and hydromagnetics, has been given by Chandrasekhar [1]. Bhatia [2] has considered the Rayleigh-Taylor instability of two viscous superposed conducting fluids in the presence of a uniform horizontal magnetic field. Sharma [3] has studied the instability of the plane interface between two oldroydian viscoelastic superposed conducting fluids in the presece of a uniform magnetic field. Bhatia and Steiner [4] have studied the problem of thermal instability of a Maxwellian viscoelastic fluid in the presence of rotation and have found that the rotation has a destabilizing effect in contrast to the stabilizing effect on Newtonian fluid. Bhatia and Steiner [5] have also considered the problem of thermal instability of a Maxwellian fluid in hydromagnetics and have found that the magnetic field has stabilizing effect on viscoelastic fluid just as in the case of Newtonian fluid. Eltayeb [6] has stuied the convective instability in a rapidly rotating oldroydian viscoelastic fluid. The medium has been considered to be non-porous in all the above studies.

When the fluid slowly percolates through the pores of a macroscopically homogeneous and isotropic porous medium, the gross effect is represented by Darcy's law according to which the usual viscous term in the equations of fluid motion is replaced by the resistance term $-(\mu/k_1)\vec{v}$, where μ is the viscosity of the fluid, k_1 the permeability of the medium and $\vec{v}$ the filter velocity of the fluid. Lapwood [7] has studied the sability of convective flow in hydromagnetics in a porous medium using Rayleigh's procedure. The Rayleigh instability of a thermal boundary layer in flow through porous medium has been considered by Wooding [8]. Oldroyd [9] proposed a theoretical model for a class of viscoelastic fluids. An experimental demonstration by Toms and Strawbridge [10] reveals that a dilute solution of methyl methacrylate in n-butyl acetate agrees well with the theoretical model of Oldroyd fluid.

The present paper attempts to study the stability of the plane interface separating two incompressible superposed rotating Oldroydian viscoelastic fluids in porous medium in presence of a variable magnetic field. The instability of such viscoelastic fluids in porous medium may find applications in geophysics.

2. Perturbation Equations

Let $T_{ij}, z_{ij}, e_{ij}, \mu, \lambda, \lambda_0(<\lambda), p, \delta_{ij}, v_i, x_i$ and $\frac{d}{dt}$ denote respectively the total stress tensor, the shear stress tensor, the rate-of-strain tensor, the viscosity, the stress relaxation time, the strain retardation time, the isotropic pressure, the Kronecker delta, the velocity vector, the position vector and the mobile operator. Then the Oldroydian viscoelastic fluid is described by the constitutive relations

$$T_{ij} = -p\delta_{ij} + z_{ij},$$

$$(i + \lambda\frac{d}{dt})\, z_{ij} = 2\mu(1 + \lambda_0\frac{d}{dt})e_{ij}, \qquad \dots (1)$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Relations of the type (1) were proposed and studied by Oldroyd [9]. Oldroyd [9] also showed that many rheological equations of state of general validity, reduce to (1) when linearized. $\lambda_0 = 0$ yields the fluid to be Maxwellian whereas $\lambda = \lambda_0 = 0$ gives the Newtonian viscous fluid.

Consider a static state in which an incompressible Oldroydian viscoelastic fluid is arranged in horizontal strata in porous medium and the pressure p and the density ρ are funtions of the vertical coordinate z only. The system is acted on by a variable horizontal magnetic field $\vec{H}(H_0(z), 0, 0)$, a uniform rotation $\vec{\Omega}(0, 0, \Omega)$ and a gravity force $\vec{g}(0, 0, -g)$. The character of the equilibrium of this initial static state is determined, as usual, by supposing that the system is slightly disturbed and then by following its further evolution.

Let $\vec{v}(u, v, w)$, $\delta\rho$, δp and $\vec{h}(h_x, h_y, h_z)$ denote respectively the perturbations in fluid velocity (0, 0, 0), fluid density ρ, fluid pressure p

and the magnetic field $\vec{H}(H_0(z), 0, 0)$. Then the linearized hydromagnetic perturbation equations of rotating Odroydian viscoelastic fluid in porous medium are

$$\frac{\rho}{\in}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial \vec{v}}{\partial t}=\left(1+\lambda\frac{\partial}{\partial t}\right)\Big[-\nabla\,\delta p+\vec{g}\,\delta\rho+\frac{2\rho}{\in}\,(\vec{v}\times\vec{\Omega})$$

$$+\frac{\mu_e}{4\pi}\{(\nabla\times\vec{h})\times\vec{H}+(\nabla\times\vec{H})\times\vec{h}\}\Big]-\frac{\mu}{k_1}\Big(1+\lambda_0\frac{\partial}{\partial t}\Big)\vec{v}, \quad \text{... (2)}$$

$$\nabla\cdot\vec{v}=0, \quad \text{... (3)}$$

$$\in\frac{\partial \vec{h}}{\partial t}=(\vec{H}\cdot\nabla)\vec{v}-(\vec{v}\cdot\nabla)\vec{H}, \quad \text{... (4)}$$

$$\nabla\cdot\vec{h}=0, \quad \text{... (5)}$$

$$\in\frac{\partial}{\partial t}\,\delta\rho=-\,w(D\rho), \quad \text{... (6)}$$

where $\in$ is the medium porosity, μ_e the magnetic permeability and $D=d/dz$. Equation (6) results from the fact that the density of every particle remains unchanged as we follow it with its motion.

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y and t is given by

$$\exp\,(ik_x x+ik_y y+nt), \quad \text{... (7)}$$

where k_x, k_y are horizontal wave numbers, $k^2=k_x^2+k_y^2$ and n is a complex constant.

For perturbations of the form (7), Eqs. (2)-(6) give

$$\frac{\rho}{\in}(1+\lambda n)nu=-\,(1+\lambda n)ik_x\,\delta p+(1+\lambda n)\frac{\mu_c}{4\pi}\,h_z\,(DH)$$

$$+\,(1+\lambda n)\,\frac{2\rho\Omega}{\in}\,v-\frac{\mu}{k_1}\,(1+\lambda_0 n)u, \quad \text{(8)}$$

$$\frac{\rho}{\in}(1+\lambda n)nv=-\,(1+\lambda n)ik_y\,\delta p+(1+\lambda n)\frac{\mu_c H}{4\pi}\,(ik_x h_y-ik_y h_x)$$

$$-\,(1+\lambda n)\,\frac{2\rho\Omega}{\in}\,u-\frac{\mu}{k_1}\,(1+\lambda_0 n)v, \quad \text{(9)}$$

$$\frac{\rho}{\in}(1+\lambda n)nw=-\,(1+\lambda n)\,[D\delta p+g\delta\rho]+(1+\lambda n)\,\frac{\mu_e H}{4\pi}.$$

$$[(ik_x h_z-Dh_x)-h_x\,\frac{DH}{H}]-\frac{\mu}{k_1}(1+\lambda_0 n)w, \quad \text{... (10)}$$

$$ik_x u+ik_y v+Dw=0, \quad \text{... (11)}$$

$$ik_x h_x+ik_y h_y+Dh_z=0, \quad \text{... (12)}$$

$$\in nh_x=ik_x Hu-w(DH), \quad \text{... (13)}$$

$$\in nh_y = ik_x Hv, \qquad \text{... (14)}$$

$$\in nh_z = ik_x Hw, \qquad \text{... (15)}$$

$$\in n\, \delta\rho = - w\, D\rho. \qquad \text{... (16)}$$

Multiplying Eqs. (8) and (9) by $by - ik_x, - ik_y$ respectively, adding and using Eqs. (11), (13)-15, we obtain

$$\frac{\rho}{\in} n(1 + \lambda n)Dw = -k^2(1 + \lambda n)\delta p - \frac{\mu}{k_1}(1 + \lambda_0 n)\, Dw$$

$$-\frac{2\rho\Omega}{\in}(1 + \lambda n)\zeta + (1 + \lambda n)k_x k_y \frac{\mu_e H^2}{4\pi n \in}\zeta + (1 + \lambda n)\frac{\mu_e H k^2}{4\pi H \in} w(DH), \qquad \text{... (17)}$$

where ζ the z-component of vorticity, is given by

$$\zeta \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = ik_x v - ik_y\, u.$$

Multipying Eqs. (8) and (9) by $-ik_y$ and $+ik_x$, respectively, adding and using Eqs. (11), (13)-(15), we obtain

$$\zeta = \frac{2(1 + \lambda n)\, \Omega\, Dw}{(1 + \lambda n)n + (1 + \lambda_0 n)\dfrac{\in \nu}{k_1} + (11 + \lambda n)\dfrac{k^2 x V_A^2}{n}}, \qquad \text{... (18)}$$

where $V_A^2 = \mu_e H^2/4\pi\rho$ is square of the Alfven velocity and $\nu(=\mu/\rho)$ stands for kinematic viscosity. Eliminating δp between Eqs. (10) and (17), using (18) and the relation

$$ik^2 u = -(k_x Dw + k_y \zeta) = -\left[k_x + \frac{2(1 + \lambda n)k_y\, \Omega\, n}{(1 + \lambda n)n^2 + (1 + \lambda_0 n)n\dfrac{\in \nu}{k_1}} + (1 + \lambda n)k_x^2 V_A^2\right] Dw, \qquad \text{... (19)}$$

we get after simplification

$$[(1 + \lambda n)n + (1 + \lambda_0 n)\frac{\in \nu}{k_i}][D(\rho DW) - k^2\rho w] + [gk^2(\frac{1 + \lambda n)}{n}]\, (D\rho)w$$

$$+ 4(1 + \lambda n)^2\, \Omega^2 n\left[\frac{\rho Dw}{(1+ \lambda n)n^2 + (1 + \lambda_0 n)n\dfrac{\in \nu}{k_1} + (1 + \lambda n)k_x^2 v_A^2}\right]$$

$$+ (1 + \lambda n)\frac{\mu_e k_x^2}{4\pi n}[D(H^2 Dw) - k^2 H^2 w] = 0. \qquad \text{... (20)}$$

3. The Case of Exponentially Varying Density, Viscosity and Magnetic Field

Assume the stratifications in density, viscosity and Assume the stratifications in density, viscosity and magnetic field of the form

$$\rho = \rho_0 e^{\beta z}, \ \mu = \mu_0 e^{\beta z}, \ H^2 = H^2 = H_0^z \, e^{\beta z}, \qquad \text{... (21)}$$

where ρ_0, μ_0, H_0 and β are constants. Equations (21) imply that the coefficient of kinematic viscosity ν and the Alfven velocity are constant every where. Using the stratification of the form (21), Eq. (20) trnasforms to

$$\left[(1+\lambda n)n + (1+\lambda_0 n)\frac{\in \nu_0}{k_1} + \frac{(1+\lambda_n)}{n} V_A^2 k_x^2 + \frac{4(1+\lambda n)^2 \Omega^2 n}{(1+\lambda n)n^2 + (1+\lambda_0 n)n\frac{\in \nu_0}{k_1} + (1+\lambda n)V_A^2 k_x^2}\right] D^2 w$$

$$+\left[(1+\lambda n)n + (1+\lambda_0 n)\frac{\in \nu_0}{k_1} + \frac{(1+\lambda n)}{n} V_A^2 k_x^2 + \frac{4(1+\lambda n)^2 \Omega^2 n}{(1+\lambda n)ns + (1+\lambda_0 n\frac{E\nu_0}{k_1} + (1+\lambda n)V_A^2 k_x^2}\right] \beta D w \qquad \text{... (22)}$$

$$-\left[(1+\lambda n)n + (1+\lambda_0 n)\frac{\in \nu_0}{k_1} + \frac{(1+\lambda n)}{n} V_A^2 k_x^2 - g\beta\frac{(1+\lambda n)}{n}\right] k^2 w = 0,$$

where $\nu_0 = \dfrac{\mu_0}{\rho_0}$ and $V_A^2 = \dfrac{\mu_e H_0^2}{4\pi\rho_0}$ are constants.

The general solution of Eq. (22) is

$$w = A_1 e^{q_1 z} + A_z e^{q_{2z}}, \qquad \text{... (23)}$$

where A_1, A_2 are two arbitrary constants and q_1, q_2 are the roots of the equation

$$\left[(1+\lambda n)n + (1+\lambda_0 n)\frac{\in \nu_0}{k_1} + \frac{(1+\lambda n)}{n} V_A^2 k_x^2 + \frac{4(1+\lambda n)^2 \Omega^2 n}{(1+\lambda n)n^2 + (1+\lambda_0 n\frac{\in \nu_0}{k_1} + (1+\lambda n)V_A^2 k_x^2}\right] q^2$$

$$+\left[(1+\lambda n)n + (1+\lambda_0 n)\frac{\in \nu_0}{k_1} + \frac{(1+\lambda n)}{n} V_A^2 k_x^2 + \frac{4(1+\lambda n)^2 \Omega^2 n}{(1+\lambda n)ns + (1+\lambda_0 n\frac{\in \nu_0}{k_1} + (1+\lambda n)V_A^2 k_x^2}\right] \beta q$$

$$-\left[(1+\lambda n)n+(1+\lambda_0 n)\frac{\in\nu_0}{k_1}+\frac{(1+\lambda n)}{n}V_A^2k_x^2-g\beta\frac{(1+\lambda n)}{n}\right]k^2=0. \qquad \dots (24)$$

If the fluid is supposed to be confined between two rigid planes at $z=0$ and $z=d$, then the vanishing of w at $z=0$ is satisfied by the choice

$$w=A\,(e^{q_1 z}-e^{q_2 z}), \qquad \dots (25)$$

while the vanishing of w at $z=d$ requires

$$\exp\,(q_1-q_2)d=1, \qquad \dots (26)$$

which imply that

$$(q_1-q_2)d=2im\pi, \qquad \dots (27)$$

where m is an integer.

Eq. (24) gives

$$q_{1,2}=\left[-\beta/2\left\{(1+\lambda n)n+(1+\lambda_0 n)\frac{\in\nu_0}{k_1}+\frac{(1+\lambda n)}{n}V_A^2k_x^2 + \frac{4(1+\lambda n)^2\Omega^2 n}{(1+\lambda n)n^2+(1+\lambda_0 n)n\frac{\in\nu_0}{k_1}+(1+\lambda n)V_A^2k_x^2}\right\}\right.$$

$$\pm\frac{1}{2}\left[betasup2\left\{(1+\lambda n)n+(1+\lambda n)\frac{\in\nu_0}{k_1}+\frac{(1+\lambda n)}{n}V_A^2k_x^2 + \frac{4(1+\lambda n)^2\Omega^2 n}{(1+\lambda n)n^2+(1+\lambda_0 n\frac{\in\nu_0}{k_1}+(1+\lambda n)V_A^2k_x^2}\right\}^2\right.$$

$$+4k^2\left\{(1+\lambda n)n+(1+\lambda_0 n)\frac{\in\nu_0}{k_1}+\frac{(1+\lambda n)}{n}V_A^2k_x^2 + \frac{4(1+\lambda n)^2\Omega^2 n}{(1+\lambda n)n^2+(1+\lambda_0 n\frac{\in\nu_0}{k_1}+(1+\lambda n)V_A^2k_x^2}\right\}$$

$$\left.\left.\cdot\left\{(1+\lambda n)n+(1+\lambda_0 n)\frac{\in\nu_0}{k_1}+\frac{(1+\lambda n)}{n}V_A^2k_x^2-g\beta\frac{(1+\lambda n)}{n}\right\}\right]^{1/2}\right].$$

$$\left[(1+\lambda n)n+(1+\lambda_0 n)\frac{\in\nu_0}{k_1}+\frac{(1+\lambda_n)}{n}V_A^2k_x^2\right.$$

$$+\frac{4(1+\lambda n)^2\Omega^2 n}{(1+\lambda n)n^2+(1+\lambda_0 n\,\frac{\in \nu_0}{k_1}+(1+\lambda n)V_A^2k_x^2}\Bigg]^{-1} \quad \ldots (28)$$

Inserting the values of q_1, q_2 from (28) in Eq. (27) and simplifying, we obtain

$$A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0 = 0, \quad \ldots (29)$$

where $\qquad A_6 = A\lambda^2, \qquad A_5 = 2A\lambda[1+\lambda_0],$

$$A_4 = [A\{1+\lambda_0^2\,\frac{\in^2 \nu_0}{k_1^2} + 2(\lambda+\lambda_0)\} + B\{\lambda^2\Omega^2\} + \lambda^2\{2k_x^2 v_A^2 v_A^2 A - c\}],$$

$$A_3 = \left[2A\{1+\lambda_0\,\frac{\in^2 \nu_0^2}{k_1^2}\} + 2B\{\lambda\Omega^2\} + \lambda\{(2+\lambda_0\,\frac{\in \nu_0}{k_1})(2k_x^2V_A^2A - C)\right],$$

$$A_2 = \left[A\left\{\frac{\in^2 \nu_0^2}{k_1^2}\right\} + B\{\Omega^2\} + (2k_x^2V_A^2A - C)\left\{1+(\lambda+\lambda_0)\,\frac{\in \nu_0}{k_1}\right\} + \lambda^2k_x^2V_A^2(k_x^2V_A^2A - C)\right]$$

$$A_1 = 2\lambda k_x^2V_A^2(k_x^2V_A^2A - C) + \frac{\in \nu_0}{k_1}\,(2k_x^2 v_A^2 A - C),$$

$$A_0 = k_x^2V_A^2\,(k_x^2V_A^2\,A - C), \quad \ldots (30)$$

where, we have put

$$A = \beta^2 d^2 + 4m^2\pi^2 + 4k^2d^2,$$
$$B = \beta^2 d^2 + 4m^2\pi^2,$$
$$C = 4k^2d^2 g\beta.$$

Eq. (29) is the dispersion relation studying the effect of rotation and the variable (exponential) horizontal magnetic field on the stability of stratified (exponentially varying density, viscosity) oldroydian viscoelastic field in porous medium.

For stable stratification $\beta < 0$, Eq. (29) does not have any change of sign and so has no positve root of n and the system is always stable for disturbances of all wave numbers.

For unstable stratification and if

$$V_A^2 < \frac{4g\beta k^2}{(\beta^2 + 4m^2\frac{\pi^2}{d^2} + 4k^2)kx^2}, \quad \ldots (31)$$

the constant term in Eq. (29) is negative, therefore, it has at least one positive real root and hence, the system is unstable for all wave numbers satisfying the inequality

$$k^2 < \frac{g\beta \sec^2\theta}{V_A^2} - \frac{\beta^2 d^2 + 4m^2\pi^2}{4d^2}, \qquad \text{... (32)}$$

where θ is the angle between k_x and k. (i.e. $k_x = k\cos\theta$).

If $\beta > 0$ (unstable stratification) and also

$$V_A^2 > \frac{4\beta k^2}{\left(\beta^2 + \frac{4m^2\pi^2}{d^2} + 4k^2\right)k_x^2}, \qquad \text{... (33)}$$

then Eq. (29) has no positive root and so the system is stable.

Thus, for unstable density stratification and magnetic field such that

$$V_A^2 > \frac{4\beta k^2}{\left(\beta^2 + \frac{4m^2\pi^2}{d^2} + 4k^2\right)k_x^2}, \qquad \text{... (34)}$$

the system is unstable for all wave unmbers satisfying

$$k^2 < g\beta\,\frac{\sec^2\theta}{V_A^2} - \frac{\beta^2 d^2 + 4m^2\pi^2}{4d^2}.$$

Also, it is clear from Eq. (29) that rotation does not affect the stability or instability , as such, of a stratification.

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