Jñānābha, Vol 24, 1994
(Dedicated to Professor J.N. Kapur on his 70th Birthday)

SOME REMARKS ON A FIXED POINT IN A-METRIC SPACE

by

A.K. Agrawal and Abrar Ahmed,
Department of Mathematics,
Sahu Jain College, Najibabad-246 763, U.P.

(Received : December 10, 1994)

ABSTRACT

In this paper, we follow Rawat and Shahu, [3] and generalise the result using Patil and Achari [2], and Dubey [1] for pair of mappings in 2-metric space.

Theorem 1 :

Let f, g be a pair of self maps of a complete metric space X. If there exist positive real numbers $0 \leq \beta\, \gamma, \delta < 1$ *satisfying* $\alpha + \beta < 1$, $\beta + \delta < 1$, and if (1) Let

$$(1)\ (d(fu_1, gu_2))^2 \leq \alpha\, d(u_1, fu_3)\, d(u_2, gu_4) + \beta d(u_1, u_2)^2 + \gamma d(u_2, gu_4)\, d(u_3, gu_6) + \delta d(u_4, fu_5)^2$$

for all u_i, $i = 1, 2, \ldots, 6 \in X$ then f, g have a common unique fixed point.

Proof : Let $x, y \in X$ and points

$$u_1 = gfx,\ u_2 = fgy,\ u_3 = gy,\ u_4 = fx,\ u_5 = x,\ u_6 = y$$

Then

$$(d(fgfx, gfgy))^2 \leq (\alpha + \beta)\,(d(gfx, fgy))^2$$

if $\alpha + \beta < 1 = k^2$, then $0 < k < 1$

$$d(fgfx, gfgy) \leq d(gfx, fgy$$

If we put $x_n = fx_{n-1}, x_{n+1} = gx_n$ for every $n = 1, 2, 3$

and $x = x_{n-3}$, $y = x_{n-2}$, then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fgf\,x_{n-3}, gfg\,x_{n-2}) \\ &\leq kd(Ifx_{n-3}, fgx_{n-2}) \\ &\leq kd(x_{n-1}, x_n) \\ &\leq k^2 d(x_{n-2}, x_{n-1}) \\ &\ldots \quad \ldots \quad \ldots \\ &\leq K^n d(x_0, x_1) \end{aligned}$$

where $K < 1$. Therefore $K^n \to 0$ as $n \to \infty$.

So the sequence defined above is Cauchy sequence in a complete metric space X. There exists a point z in X such that $x_n \to z, \ n \to \infty$

Let $u_1 = u_3 = u_5 = z, \quad u_2 = u_4 = u_6 = x_n$

$$(d(fz, x_{n+1}))^2 = (d(fz, gx_n))^2 \leq \alpha d(z, fz)\, d(x_n, gx_n) + \beta d(z, x_n)^2$$
$$+ \gamma d(x_n, gx_n) d(z, gx_n) + \delta d(x_n, fz)^2$$

$n \to \infty$

$$(d(fz, z))^2 \leq \alpha d(z, fz) d\,(x_n, x_{n+1}) + \beta d(z, x_n)^2$$
$$+ \gamma d(x_n, x_{n+1}) d(z, x_{n+1}) + \delta d(x_n, fz)^2,$$

$n \to \infty$

$$(d(fz, z))^2 \leq \alpha.0 + \beta.0 + \gamma.0 + \delta d(z, fz)^2,$$
$$(d(fz, z))^2 \leq \delta (d(z, fz))^2.$$

A contradiction $0 < \delta < 1$.

Hence $fz = z$ i.e. z is a fixed point of f. Similarly z is also a fixed of g.

To claim the uniqueness we say that

$$(d(z, w))^2 = (d(fz, gw))^2$$
$$\leq \alpha d(z, fz) d(w, gw) + \beta d(z, w)^2 + \gamma d(w, gw)\, d(z, gw) + \delta d(w, fz)^2$$
$$d(z, w)^2 \leq (\beta + \delta)\, d(z, w)^2$$
$$\beta + \delta < 1,$$

which is a contradiction. Hence $z = w$.

Theorem 2 :

If $f = g$, we get, which we give without proof.

$$(d(fu_1, fu_2))^2 \leq \alpha\, d(u_1, fu_3) d\,(u_2, fu_4) + \beta d(u_1, u_2)^3$$
$$+ \gamma d(u_2, fu_4) d(u_3, fu_6) + \delta d(u_4, fu_5)^2$$

$0 < u_1, u_2, \ldots, u_6 \in X < 1$ then f has a fixed point. We state and prove the following theorem which is direct application of the theorem 1 for a family of mappings.

Theorem 3 :

Let $\{f_k\}$ $(k = 1, 2, 3, \ldots, n)$ be a family of mapping of a complete metric space X into itself. If f_k satisfies the conditions

(*i*) $f_1 f_2 \cdots f_n$. commute with every f_k.

(*ii*) $(d(f_1 f_2 \cdots f_n u_1, f_{n-1} \cdots f_2 f_1 u_2))^2$

$$\leq \alpha d(u_1, f_1 f_2 \dots f_n u_3) d(u_2, f_n f_{n-1} \dots f_2 f_1 u_4) + \beta d(u_1, u_2)^2$$
$$+ \gamma d(u_2, f_n f_{n-1} \dots f_2 f_1 u_4)\, d(u_3, f_n f_{n-1} \dots f_1 u_6)$$
$$+ \delta d(u_4, f_1 f_2 \dots f_n u_5)^2$$

$u_1 \dots u_6$ in X and $0 \leq \alpha, \beta \dots < 1$ then $\{f_k\}$ have a common fixed point.

Proof :

From (*ii*) If we put $f_1 \dots f_n = f, \quad f_n \dots f_1 = g$

Then it takes the form (1) by theorem 1 f, g have a unique fized point of z. *i.e.* $fz = z = gz$ for any $f_k, f_k(z) = f_k z$ by a view of (1)

$f(k, k\, z) = f_k\, z$ so $f_k(z)$ is fized point of f and z is a fized point of z. By putting $u_1 = u_3 = u_5 = fkz,\ u_2 = u_4 = u_6 = z$ in (1)

$$d(f_k z, z)^2 = d(ff_k z, gz)^2$$
$$\leq \alpha d(f_k z, ff_k z)\, d(z, gz) + \beta d(f_k z, z)^2 + \gamma d(z, gz)\, d(f_k z, gz) + \delta d(z, ff_k z)^2$$
$$d(f_k z, z)^2 \leq (\beta + \delta) d(z, f_k z)^2, \quad (\beta + \delta < 1).$$
$$d(f_k z, z)^2 \leq (\beta + \delta)\, d(f_k z, z)^2.$$

A contradiction. Hence $fkz = z$ i.e. z is a fixed point of a family of mappings $\{f_k\}$.

REFERENCES

[1] Praveen Dubey, A result on common fixed point and sequence of mappings, *Garula*, **2** (1991), 134.

[2] P.T. Patil and J. Achari. A note on common fized points in 2-metric spaces, *The mathematics Education*, **22** (1988), 126, 129.

[3] Pratima Rawat and P.L. Sahu Common fized point theorem for 3-mappings, *Acta Sciencia Indica*, XVM 4(1989, 351.