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COMMON FIXED POINT THEOREMS FOR DENSIFYING MAPPINGS

By

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ABSTRACT

Several fixed point theorems for multivalued compatible maps are proved. As corollaries we obtain several known fixed point theorems for densifying maps.

In a recent paper Diviccaro, Khan and Sessa [4] established the following theorem.

Theorem 1. *Let f and g be two densifying and commuting maps of a bounded complete metric space (X, d) such that*

(1) $d(fx, fy) < \max\{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}$

for all x, y in X such that the right hand side of (1) is positive and $fX \subseteq gX$. Then f and g have a unique common fixed point.

The purpose of this note is to prove the aboe result by replacing the hypothesis of commuting by the weaker assumption of compatibility. This goal will be realized by first proving some fixed point theorems for multivalued maps, and then obtaining the desired result as a corollary.

We first state some familiar definitions. For a metric space (X, d), f a selfmap of X, A a bounded subset of X, $\alpha(A)$ denotes the measure of monocompactness of A; i.e., the infimum of all $\varepsilon > 0$ such that A admits a finite covering consisting of subsets with diameters less than ε.

The following well-known properties of α hold :

(*i*) $0 \leq \alpha(A) \leq \text{diam } A$,

(*ii*) $\alpha(A) = 0$ iff A is precompact,

(*iii*) $\alpha(A \cup B)$ max max $\{\alpha(A), \alpha(B)\}$ for any bounded subsets A, B of X,

(*iv*) $A \subseteq B$ implies that $\alpha(A) \leq \alpha(B)$.

f is said to be densifying if f is continuous and, for any bounded non precompact subset A of X, we have $\alpha(f(A)) < \alpha(A)$.

Let $B(X)$ denote the set of bounded subsets of a complete metric space (X, d) and define a function $\delta : B(X) \times B(X) \to [0, \infty)$ by $\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}$. It is immediate that δ satisfies the triangular inequality and that $\delta(A, B) = 0$ iff $A = B = \{a\}$. Let I be a selfmap of X, $F : X \to B(X)$. F and I are said to be δ-compatible iff $x \in B(X)$ for $x \in X$ and $\delta(IFx_n, FIx_n) \to 0$ whenever $\{x_n\}$ is a sequence such that $Ix_n \to t$ and $\{Fx_n\} \to \{t\}$ for some t in X. (See, e.g., Jungck and Rhoades [6]).

Theorem 2. *Let F be a continuous mapping of a complete metric space (X, d) into $B(X)$, I a continuous selfmap of X Such that*

(2) $\delta(Fx, Fy) \le c \max \{d(Ix, Iy), \delta(Ix, Fx), \delta(Iy, Fy), \delta(Ix, Fy), \delta(Iy, Fx)\}$

for all x, y in X, where $0 \le c \le 1$. If $FX \subseteq IX$ and F and I are δ-compatible, then F and I have a unique common fixed point z and further $Fz = \{z\}$.

Proof. The proof follows along the lines of the proof of theorem 1 of Fisher [5]. One defines a sequence $\{x_n\}$ by $x_0 \in X, y_1 \in Fx_0$. Since $FX \subseteq IX$, choose x_1 so that $Ix_1 = y_1$. In general x_n is chosen so that $Ix_n = y_n, y_n \in Fx_{n-1}$. Then, as in Fisher, it follows that Ix_n converges to a point z, and the sets $\{Fx_n\}$ converge to the set $\{z\}$.

From (2),

$$\delta(FIx_n, Fx_n) \le c \max \{\delta(I^2x_n, Ix_n), \delta(I^2x_n, FIx_n), \delta(Ix_n, Fx_n), \delta(I^2x_n, Fx_n), \delta(Ix_n, FIx_n)\}$$

$$\le c \max \{\delta(IFx_{n-1}, Ix_n), \delta(IFx_{n-1}, FIx_n) \delta(Ix_n, Fx_n), \delta(IFx_{n-1}, Fx_n), \delta(Ix_n, FIx_n)\}. \tag{3}$$

From the definition of δ-compatibility, $\delta(IFx_{n-1}, FIx_n) \to 0$. Since F and I are continuous, it then follows that $\lim \delta(IFx_{n-1}, Ix_n) = \lim \delta(FIx_n, Ix_n) = \delta(Fz, z)$ and $\lim \delta(IFx_{n-1}, Fx_n) = \lim \delta (FIx_n, Fx_n) = \delta(Fz, z)$. Taking the limit of (3) as $n \to \infty$, we obtain $\delta(Fz, z) \le c \max \{\delta(Fz, z), \delta(Fz, Fz)\} = \delta(Fz, Fz)$. Again using (2),

$$\delta(FIx_n, FIx_n) \le c \max \{d(I^2x_n, I^2x_n), \delta(I^2x_n, FIx_n)\}.$$

Taking the limit as $n \to \infty$ yields $\delta(Fz, Fz) \le c\delta (Fz, Fz)$, from which it follows that Fz is the singleton set $\{z\}$.

Since $FX \subseteq IX$, there exists a point w such that $Iw = z$. Using (2),

$$\delta(Fx_n, Fw) \le c \max \{d(Ix_n, Iw), \delta (Ix_n, Fx_n), \delta(Iw, Fw), \delta(Ix_n, Fw), \delta(Iw, Fx_n)\}.$$

Taking the limit as $n \to \infty$ gives $\delta(z, Fw) \le c\delta(z, Fw)$, which implies that $Fz = \{z\} = Iw$. From the δ-compatibility of I and F it follows that

$\{z\} = Fz = FIw = IFw = \{Iz\}$, and z is a fixed point of I. Condition (2) forces the common fixed point to be unique.

Theorem 3. *Let F be a continuous mapping of a compact metric space (X, d) into $B(X)$, I a continuous selfmap of X such that*

(4) $\delta(Fx, Fy)\,\delta(Fx, Fy) < \max\{d(Ix, Iy), \delta(Ix, Fx), \delta(Iy, Fy), \delta(Ix, Fy), \delta(Iy, Fx)\}$

for all x, y in X for which the right-had-side of (4) is positive. If $FX \subseteq IX$ and F and I are δ-compatible, then F and I have a unique common fixed point z and further $Fz = \{z\}$.

Proof. Suppose there exists a $0 \le c < 1$ such that F and I satisfy inequality (2) for all x, y in X for which the right had side of inequality (2) is positive. If the right-hand-side of (2) is zero it follows that we must have $Fx = Fy = \{Ix\} = \{Iy\}$, which forces the left had side of (2) to be zero, so (2) is satisfied for all x, y in X and the result follows from Theorem 2.

If no such number c exists, then there is an increasing sequence $\{c_n\}$, with limit 1, and sequences $\{x_n\}$, $\{y_n\}$ such that

$$d(fx_n, Fy_n) \ge c_n \max\{d(Ix_1, Iy_n), \delta(Ix_n, Fx_n), \delta(Iy_n, Fy_n), \delta(Ix_n, Fy_n), \delta(Iy_n, Fx_n)\}.$$

Since X is compact, we may assume, without loss of generality, that the sequecnes $\{x_n\}$ and $\{y_n\}$ converge to points x and y respectively. Letterring n tend to infinity in the above inequality leads to, since F and I are continuous,

$$\delta(Fx, Fy) \ge \max\{d(Ix, Iy), \delta(Ix, Fx), \delta(Iy, Fy), \delta(Ix, Fy), \delta(Iy, Fx)\}:$$

Using the above inequality, along with (4) it follows that we must have $Fx = Fy = \{Ix\} = \{Iy\}$. From the δ-compatibility of I and F, we then have $FIx = IFx$. Thus $F^2x = FIx = FIx = IFx = \{I^2x\}$.

Suppose that $FIx \ne Fx$. Then, from (4),

$$\delta(FIx, Fx) < \max\{d(I^2x, Ix), \delta(I^2x, FIx), \delta(Ix, Fx), \delta(I^2x, Fx), \delta(Ix, FIx)\}$$
$$= \delta(FIx, Fx),$$

a contradiction. Therefore $FIx = Fx$, and Ix is a fixed point of F. With $Ix = z$ it follows that $\{Iz\} = \{I^2x\} = FIx = Ix = z$, and z is also a fixed point of I. Definition (4) forces uniqueness of the common fixed point.

Corollary 1. *Let f and g be continuous selfmaps of a bounded metric space (X, d) satisfying inequality (1) for all x, y in X for which the right-hand-side of (1) is positive. If $fX \subseteq gX$ and f and g are compatible, then f and g have a unique common fixed point z.*

Proof. Define $A = O(x_0)$, the orbit of x, for some x_0 in X. Since X is bounded, so is A. Moreover, $A = \{x_0\} \cup \{f(A)\} \cup \{g(A)\}$. Thus $\alpha(A) = \max\{(f(A), \alpha(g(A))\}$. Since f and g are densifying and X is complete, it follows that A is compact. Now apply Theorem 3.

Corollary 2. *Let (X, d) be a complete metric space, f a densifying selfmap of X satisfying*

$$d(fx, fy) < \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}$$

for each x, y *in* $X, x \neq y$. *If, for some* x_0 *in* X, *the sequence* $\{x_n\}$ *defined by* $\{x_0, x_1 = f(x_0), x_2 = f(x_1), \ldots\}$ *is bounded, then* f *has a unique fixed point.*

Proof. With $A = O(x)$, A is bounded, and the result follows from Corollary 1 by setting $g = f$.

Theorem 4. *Let* f, g *be selfmaps of a complete metric space* (X, d), f *possessing a unique fixed point* z *and* g *commuting with* f. *Then* z *is the unique common fixed point of* f *and* g.

Proof. $gz = gfz = fgz$, and gz is also a fixed point of f. Since the fixed point of f is unique, $gz = z$.. Suppose that w is also a common fixed point of f and g. Then w is a fixed point of f. Since f has a unique fixed point, $w = z$.

Remarks. 1. Theorems 1, 2, and Corollary 2 of Fisher [5] are special cases of Theorems 2,3, and Corollary 1, respectively, of this paper.

2. Theorem 4 of Diviccaro, Khan and Sessa [4] is a special case of Corollary 1.

3. Theorem 1 of Achari [1], Theorem 3 of Chatterjee [2], and Theorem 1 of Chattopadhyay [3] are special cases of Corollary 2.

4. Theorem 2 of Achari [1] is a special case of Theorem 4.

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