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(Dedicated to Professor J.N. Kapur on his 70th Birthday)

TWO FIXED POINT THEOREMS FOR NONSELF MAPPINGS

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ABSTRACT

Nadim Assad [1] established two fixed point theorems for nonself maps. In this paper we generalize his contractive definitions and establish the corresponding fixed point theorems.

Nadim Assad [1] established a fixed point theorem for a nonself map satisfying the condition

(1) $d(Tx, Ty) \leq b(x, y)\,[d(x, Tx) + d(y, Ty)] + c(x, y) \min\{d(x, Ty), d(y, Tx)\}$,

where b and c are decreasing functions from $R^+ \to [0, 1)$ such that $2b(t) + c(t) < 1$ for all $t > 0$. He also established a fixed point theorem for K compact, T continuous, and strict inequality in (1) with $2b(t) + c(t) = 1$.

The purpose of this paper is to generalize Assad's definition [1], and then prove the corresponding fixed point theorems.

Let $x_0 \in X, x_1' := Tx_0$ If $x_1' \in K$, set $x_1' = x_1$. Otherwise, choose $x_1 \in \partial K$ such that $d(x_0, x_1') = d(x_0, x_1) + d(x_1, x'_1)$. Then $x_1 \in K$ and Tx_1 is defined. Suppose that, for $n \geq 1$, $\{x_0, x_{1,} \ldots\}$ and $\{x_0', x_1', \ldots\}$ have been chosen so that, for $1 \leq i \leq n$,

(i) $x_i' = Tx_{i-1}$,

(ii) $x'_i = x_i$ if $x_i' \in K$, or

(iii) $x_i \in \partial K$ and satisfies the relation

$$d(x_{i-1}, x_i') \ldots = d(x_{i-1}, x_i) + d(x_i, x_i').$$

Let $x'_{n+1} = Tx_n$. If $x'_{n+1} \in K$, set $x_{n+1} = x'_{n+1}$. Otherwise choose $x_{n+1} \in \partial K$ so that $d(x_n, x'_{n+1}) = d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1})$. The sequecne $\{x_n\}$ so constructed shall be called the general orbit of T at x_0.

Let $P := \{x_i \in \{x_n\} : x_i = x_i'\}$ and $Q := \{x_i \in \{x_n\} : x_i \neq x_i'\}$. Note that, if $x_n \in Q$, then $x_{n+1} \in P$.

Theorem 1. *Let X be a Banach space, K a nomempty closed subset of X, $T : K \to X$ satisfying the condition that $x \in \partial K$ implies that $Tx \in K$ and, for all $x, y \in X$,*

(2) $$d(Tx, Ty) \leq b(x, y) \max \{d(x, Tx), d(y, Ty)\} + c(x, y) \min \{d(x, Ty), d(y, Tx)\}$$

where b and c are decreasing functions from $R^+ \to [0, 1)$ such that $b(t) + c(t) < 1$ for all $t > 0$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$, $\{x_n\}$ a general orbit of x_0, $\tau_n := d(x_n, x_{n+1})$. Without any loss of generality we may assume that $\tau_n > 0$ for each n. For, if there exists a value of n such that $\tau_n = 0$, then $x_n = x_{n+1}$. If $x_n \in \partial K$, then $x'_{n+1} \in K$ so that $x'_{n+1} = x_{n+1}$. Thus $x_n = x_{n+1} = Tx_n$ and x_n is a fixed point of T. If $x_n \notin \partial K$, then $x_n \in K$ and either $x'_{n+1} \in K$ or $x'_{n+1} \notin K$. If $x'_{n+1} \in K$, then, as before, $x_n = x_{n+1} = Tx_n$, and x_n is a fixed point of T. If $x'_{n+1} \notin K$, then there exists an $x_{n+1} \in \partial K$ such that $d(x_n, x_{n+1})\, d(x_{n+1}, x'_{n+1})$, and $x_n = x_{n+1}$ is impossible.

We shall first show that $\tau_{n+1}, < \max \{\tau_n, \tau_{n-1}\}$.

Case (*a*). Suppose that $x_{n+1}, x_{n+2} \in P$. Then, from (2),

$$\tau_{n+1} = d(x_{n+1}, x_{n+2}) \leq b(d(x_n, x_{n+1})) \max \{d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\}$$
$$+ c(d(x_n, x_{n+1}) \min \{d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n)\}$$

and $\tau_{n+1} < \tau_n$.

Case (*b*). Suppose that $x_{n+1} \in P, x_{n+2} \in Q$. Then $d(x_{n+1}, x'_{n+2}) = d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x'_{n+2})$, which implies that

$$\tau_{n+1} < d(x_{n+1}, x'_{n+2}) = d(x'_{n+1}, x'_{n+2}) = d(Tx_n\, Tx_{n+1})$$
$$\leq b(d(x_n, x_{n+1}) \max \{d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\}$$
$$+ c(d(x_n, x_{n+1})) \min \{d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n)\}$$
$$= b(d(x_n, x_{n+1}) \max \{\tau_n, d(x_{n+1}, x'_{n+2})\},$$

which implies that $\tau_{n+1} < \tau_n$.

Case (*c*). Suppose that $x_{n+1} \in Q, x_{n+2} \in P$. Since no two consecutive $x'_n s$ can lie in Q, $x_n \in P$.

Proposition 1. *[1] Let $x, y \in X, X$ a Banach space, and $\mu := \lambda x + (1 - \lambda)y$, where $0 \leq \lambda \leq 1$. Then, for any $w \in X$, $\|w - \mu\| \leq$ max $\{\|w - x\|, \|w - y\|\}$.*

Using Proposition 1, $\tau_{n+1} \leq \max \{d(x'_{n+1}, x_{n+2}, d(x_n, x_{n+2})\}$. Suppose that the maximum is $d(x'_{n+1}, x_{n+2})$.

From (1) it follows that

$$\tau_{n+1} \leq d(x_{n+1}, x'_{n+2}) = d(x'_{n+1}, x'_{n+2}) = d(Tx_n, Tx_{n+1})$$
$$\leq b(d(x_n, x_{n+1}) \max \{d(x_n Tx_n), d(x_{n+1}\, Tx_{n+1})\}$$
$$+ c(d(x_n, x_{n+1})) \min \{d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n)\}$$
$$= b(\tau_n) \max \{d(x_n, x'_{n+1}), \tau_{n+1}\} + c(\tau_n) \min \{d(x_n, x_{n+2}), d(x_{n+1}, x'_{n+1})\}$$
$$\leq b(\tau_n) \max \{d(x_n\, x'_{n+1}), \tau_{n+1}\} + c(\tau_n)\, d(x_n, x_{n+2})$$
$$\leq b(\tau_n) \max \{d(x_n, x'_{n+1}), \tau_{n+1}\} + c(\tau_n)\, d(x'n+1, x_{x+2}).$$

If $\max \{d(x_n, x'_{n+1}), \tau_{n+1}\} = d(x_n, x'_{n+1})$, then

$$d(x'_{n+1}, x_{n+1}) \leq b(\tau_n)\, d(x_n, x'_{n+1}) + c(\tau_n)\, d(x'_{n+1}, x_{n+2}),$$

or

$$\tau_{n+1} \leq \frac{b(\tau_n)}{1 - c(\tau_n)}\, d(x_n, x'_{n+1}) < \tau_{n-1}, \text{ since } x_n \in P, x_{n+1} \in Q.$$

If $\max \{d(x_n, x'_{n+1}, \tau_{n+1}\} = \tau_{n+1}$, then

(3) $$d(x'_{n+1}, x_{n+2}) \leq b(\tau_n)\tau_{n+1} + c(\tau_n)\, d(x'_{n+1}, x_{n+2})$$

or

$$\tau_{n+1} \leq \frac{b(\tau_n)}{1 - c(\tau_n)}\, \tau_{n+1},$$

a contradiction

Suppose now that $\max \{d(x'_{n+1}; x_{n+2}), d(x_n, x_{n+2})\}$ is $d(x_n, x_{n+2})$. Then

$$\tau_{n+1} \leq d(x_n, x_{n+2}) = d(Tx_{n-1}, Tx_{n+1})$$
$$\leq b(d(x_{n-1}, x_{n+1}) \max \{d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\}$$
$$c(d(x_{n-1}, x_{n+1})) \min \{d(x_{n-1}, Tx_{n+1}), d(x_{n+1}, Tx_{n-1})\}$$

(4) $$\leq b(d(x_{n-1}, x_{n+1})) \max \{\tau_{n-1}, \tau_{n+1}\} + c(d(x_{n-1}, x_{n+1}))\tau_n.$$

If $\max \{\tau_{n-1}, \tau_{n+1}\} = \tau_{n-1}$, then we have

$$\tau_{n+1} \leq b(d(x_{n-1}, x_{n+1}))\, \tau_{n-1} + c(d(x_{n-1}, x_{x+1}))\tau_n$$
$$< [b(d(d(x_{n-1}, x_{n+1})) + c(d(x_{n-1}, x_{n+1}))]\tau_{n-1} < \tau_{n-1},$$

since $x_{n-1}, x_n \in P$.

If $\max \{\tau_{n-1}, \tau_{n+1}\} = \tau_{n+1}$, then

$$\tau_{n+1} \leq \frac{c(d(x_{n-1}, x_{n+1}))}{1 - b(d(x_{n-1}, x_{n+1}))}\, \tau_n < \tau_{n-1}.$$

Therefore, in all cases, $\tau_{n+1} < \max\{\tau_n, \tau_{n-1}\}$.

The remaining cases, $x_{n+1}, x_{n+2} \in Q$, cannot occur.

Next we shall show that $\{\tau_n\}$ converges to zero. There are two possibilities. Either $\{x_n\}$ possesses a subsequence $\{x_{n(k)}\}$ with the property that $x_{n(k)+1}, x_{n(k)+2} \in P$, or it doesn't.

Suppose such a sequence exists.

Fact 1. $\tau_{n+1} < max\,\{\tau_n, \tau_{n-1}\}$ *implies that* $\tau_{n+1} \le max\,\{\tau_{n-k}, \tau_{n-k-1}\}$ *for each* $0 \le k \le n$.

Proof. The result is trivially true for $k = 0$. Assume the induction hypothesis. Since

$$\tau_{n-k} \le \max\{\tau_{n-k-1}, \tau_{n-k-2}\}$$

$$\tau_{n+1} < \max\{\tau_{n-k}, \tau_{n-k-1}\} < \max\{\max\{\tau_{n-k-1}, \tau_{n-k-2}\}, \tau_{n-k-1}\}$$
$$= \max\{\tau_{n-k-1}, \tau_{n-k-2}\}.$$

It then follows that

(5) $$\tau_{n(k)} \le \max\{\tau_{n(k-1)+1}, \tau_{n(k-1)+2}\}.$$

Consider $\tau_{n(k-1)+1}$. Since $x_{n(k-1)+1}, x_{n(k-1)+2} \in P$, $\tau_{n(k-1)+1} < \tau_{n(k-1)}$. For $\tau_{n(k-1)+2}$, since $x_{n(k-1)+2} \in P$, if $x_{n(k-1)+3} \in P$, then $\tau_{n(k-1)_{+2}} < \tau_{n(k-1)+1}$ by Case (a). If $x_{n(k-1)+3} \in Q$, then $\tau_{n(k-1)+2} < \tau_{n(k-1)+1}$ by Case (b). Therefore $\tau_{n(k)} < \tau_{n(k-1)}$, and $\{\tau_n\}$ converges. Call the limit τ.

Suppose $\tau > 0$. From (5),

(6) $$\tau_{n(k+1)} \le \tau_{n(k)+1} = d(x_{n(k)+1}, x_{n(k)+2}) \le \tau_{n(k)}.$$

Therefore $\lim_k d(x_{n(k)+1}, x_{n(k)+2}) = \tau$.

From Case (a), $d(x_{n(k)+1}, x_{n(k)+2} \le b(\tau_{n(k)})\,\tau_{n(k)}$. Since $\tau_{n(k)} \ge \tau$, and b is decreasing, $\tau_{n(k)+1} \le b(\tau)\tau_{n(k)}$. Taking the limit as $n \to \infty$ yields $\tau \le b(\tau)\tau$, a contradiction. Therefore $\tau = 0$.

Fact 2. *For each* j *sufficiently large there exists* $k = k(j)$ *such that* $n(k) \le j \le n(k+1)$. *Using Fact* 1, $\tau_j \le \tau_{n(k)}$.

Proof. By induction. If $j = n(k)$, then $\tau_j = \tau_{n(k)}$. If $j = n(k)+1$, then $\tau_j < \tau_{j-1} = \tau_{n(k)}$ by Case (a). If $j = n(k)+2$, then $\tau_j < \max\{\tau_{n(k)+1}, \tau_{n(k)}\}$ by what we have just provided.

Suppose that the result is true for $j = n(k)+i$. Then $j+1 \le n(k+1)$. If $j+1 = n(k+1)$, then we have equality. If $j+1 < n(k+1)$, then

$$\tau_{j+1} < \max\{\tau_{n(k)+i}, \tau_{n(k)+i-1}\} < \max\{\tau_{n(k)}, \tau_{n(k)}\} = \tau_{n(k)},$$

by the induction hypothesis.

Therefore $\lim \tau_j = 0$.

Now suppose that no such sequence exists. Then, for all n sufficiently large, one must have, for two consecutive values of n, $x_n \in P$ and $x_{n+1} \in Q$..

Let $\{x_{n(i)}\}$ be the subsequence of $\{x_n\}$ consisting of points in Q. Since no two consecutive points can be in Q, $x_{n(i)+1} \in P$. Since no two consecutive points are in P, $x_{n(i)+2} \in Q$; i.e., $n(i) + 2 = n(i + 1)$. Hence , also, $n(i - 1) = n(i) - 2$. Thus the subsequence $\{x_{n(i)}\}$ is either $\{x_{2n}\}$ or $\{x_{2n+1}\}$.

Suppose $\{x_{2n}\} \subset Q$. Then $x_{2n+1} \in P$ and, by Case (c), $\tau_{2n} < \tau_{2n-2}$. Therefore $\{\tau_{2n}\}$ is monotone decreasing and has a limit. Call it τ.

For each n, by Proposition 1,

$$\tau_{2n} = d(x_{2n}, x_{2n+1}) \le \max\{d(x_{2n-1}, x_{2n+1}), d(x'_{2n}, x_{2n+1})\}.$$

Thus there must exist at least one infinite subsequence of $\{2n\}$ for which either

(7) $$\tau_{2n(h)} \le d(x_{2n(h)-1}, x_{2n(h)+1})$$

or

(8) $$\tau_{2n(h)} \le d(x'_{2n(h)}, x_{2n(h)+1}).$$

If (8) is true, then, by Case (c),

(9) $$\tau_{2n(h)} \le d(x'_{2n(h)+1}) < \tau_{2n(h)-2}.$$

If $\tau > 0$, from the first part of Case (c),

$$\tau_{2n(h)} \le d(x'_{2n(h)-1}, x'_{2n(h)}) \le b(\tau_{2n(h)-2})\, \tau_{2n(h)-2}.$$

from Case (b). For all h sufficiently large, $\tau_{2n(h)-1} > \tau/2$. Thus, for each such h,

$$\tau_{2n(h)} \le b(\tau/2)\, \tau_{2n(h)-1}.$$

Taking the limit as $h \to \infty$ yields $\tau \le b(\tau/2)\tau$, a contradiction. Therefore $\tau = 0$. By Fact 2, $\lim \tau_j = 0$.

If (7) is satisfied, the, by the second part of Case (c),

(10) $$\tau_{2n(h)} < [b(d(x_{2n(h)-2}, x_{2n(h)}) + c(d(x_{2n(h)-2}, x_{2n(h)})]\tau_{2n(h)-2}.$$

The sequence $\{d(x_{2n(h)-2}, x_{2n(h)})\}$ is bounded. Hence it has a convergent subsequence. Without loss of generality we may assume that $\lim_h (d(x_{2n(h)-2}, x_{2n(h)}) = \rho$. Also, $\{\tau_{2n(h)}\}$ is monotone decreasing, hence convergent. Call the limit τ.

If $\rho > 0$, choose h so large that $(d(x_{2n(h)-2}, x_{2n(h)}) > \rho/2$. Then taking the limit of (10) as $h \to \infty$ yields

$$\tau \le [b(\rho/2) + c(\rho/2)]\ \tau < \tau,$$

a contradiction. Therefore $\rho = 0$.

Finally,

$$\tau_{2n(h)-2} - d(x_{2n(h)-2}, x_{2n(h)}) \le d(x_{2n(h)-1}, x_{2n(h)}) < d(x_{2n(h)-1}, x'_{2n(h)}) < \tau_{2n(h)-2}.$$

Hence $\lim_h d(x_{2n(h)-1}, x_{2n(h)})$.

Suppose $\{x_{2n+1}\} \subset Q$. Then $x_{2n+1} \in P$ for all n sufficiently large. By Case (c), $\tau_{2n+1} < \tau_{2n-1}$ and $\{\tau_{2n+1}\}$ is monotone decreasing and has a limit $\tau \ge 0$.

For each n, by Proposition 1,

$$\tau_{2n+1} = d(x_{2n+1}, x_{2n+2} \le \max\ \{d(x_{2n}, x'_{2n+2}), d(x'_{2n+1}, x_{2n+2})\}$$

Then there exists at least one infinite subsequence of $\{2n + 1\}$ for which either

(11) $\quad \tau_{2n(h)+1} \le d(x_{2n(h)}, x_{2n(h)+2})$

or

(12) $\quad \tau_{2n(h)+1} \le d(x'_{2n(h)+1}, x_{2n(h)+2})$.

If (12) holds, then, by Case (c), $\tau_{2n(h)+1} < \tau_{2n(h)-1}$,

and it follows that $\lim \tau_n = 0$.

If (11) holds then, as in case (5), $\lim \tau_n = 0$.

We shall now show that $\{x_n\}$ is Cauchy.

Suppose it is not Cauchy. Then there exist an $\varepsilon > 0$ and two subsequences $\{p(n)\}$ and $\{q(n)\}$ such that, for all n, $p(n) > q(n) \ge n$, $d(x_{p(n),} x_{q(n)}) \ge \varepsilon$ and $d(x_{p(n)-1}, x_{q(n)}) < \varepsilon$.

Proof. The first inequality is true by the negation of the definition of a Cauchy sequence. The second inequality follows by the following argument. Since $\{\tau_n\}$ is monotone decreasing with limit zero, there exists an N such that $n > N$ implies that $\tau_n < \varepsilon$. Let n be the smallest integer for which $p(n) > q(n) \ge N$. Then $p(n) > q(n) + 1$, for $p(n) = q(n) + 1$ implies that $d(x_{p(n)}, x_{q(n)}) < \varepsilon$, a contradiction. Since $d(x_{p(n)+1}, x_{q(n)}) < \varepsilon$, choose the smallest integer $r > p(n) + 1$ such that $d(x_r, x_{q(n)}) \ge \varepsilon$. Then, with $r := p(n)$, we have $d(x_{p(n)}, x_{q(n)}) \ge \varepsilon$ and $d(x_{p(n)-1}, x_{q(n)}) < \varepsilon$. Now empoly the same argument to $p(n + 1)$, $q(n + 1)$, etc.We then have, with $S_n := d(x_{p(n)}, x_{q(n)})$,

$$\varepsilon \le s_n \le d(x_{p(n)}, x_{q(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) < \tau_{p(n)}) + \varepsilon\ .$$

Taking the limit as $n \to \infty$, yields $\lim d(x_{p(n)-1}, x_{q(n)}) = \varepsilon$.

Using the triangular inequality,

$$- \tau_{p(n)} - \tau_{q(n)} + x_n \leq d(x_{p(n)+1}, x_{q(n)+1}) \leq \tau_{p(n)} + \tau_{q(n)} + s_n,$$

$$- \tau_{p(n)} + s_n \leq d(x_{p(n)+1}, x_{q(n)}) \leq \tau_{p(n)} + s_n,$$

$$- \tau_{q(n)-1} + s_n \leq d(x_{p(n),} x_{q(n)-1}) \leq s_n + \tau_{q(n)-1},$$

$$- \tau_{p(n)-1} - \tau_{q(n)-1} + s_n \leq d(x_{p(n)-1}, x_{q(n)-1}) \leq s_n + \tau_{q(n)-1} + \tau_{p(n)-1}.$$

Taking the limit as $n \to \infty$ in each of the above inequalties yields

$\lim d(x_{p(n)+1}, x_{q(n)+1}) = \lim d(x_{p(n)+1}, x_{q(n)}) = \lim d(x_{p(n)}, x_{q(n)-1})$

$= \lim d\ (x_{p(n)-1}, x_{q(n)-1}) = \varepsilon.$

Also,

$$s_n \leq d(x_{p(n)}, x_{q(n)+1}) + \tau_{q(n)} \leq s_n + 2\tau_{q(n)}$$

and $\quad s_n \leq d(x_{(p(n)-1}, x_{q(n)}) + \tau_{p(n)-1}) \leq s_n + 2\tau_{p(n)-1}$

$$= \lim d(x_{p(n)-1}, x_{q(n)}) = \varepsilon.$$

Case (A). Suppose that $x_{p(n)+1}, x_{q(n)+1} \in P$. Then

$$d(x_{p(n)+1}, x_{q(n)+1}) \quad d(Tx_{p(n)}, Tx_{q(n)})$$

$$\leq b(d)x_{p(n)}, x_{q(n)})) \max \{d(x_{p(n),} Tx_{p(n)}), d(x_{q(n)}, Tx_{q(n)}) \}$$

$$+ c(d(x_{p(n)}, x_{q(n)})) \min \{d(x_{p(n)}, Tx_{q(n)}, d(x_{q(n)}, Tx_{p(n)})\}$$

$$\leq b(\varepsilon) \max \{\tau_{p(n)}, \tau_{q(n)}\} + c(\varepsilon) \min \{d(x_{p(n)}, x_{q(n)+1}), d(x_{q(n),} x_{p(n)+1})\}.$$

Case (B). $x_{p(n)+1} \in P, x_{q(n)+1} \in Q$. Then $x_{q(n)} \in P$, and

$$d(x_{p(n)+1}, x_{q(n)}) = d(Tx_{p(n)}, Tx_{q(n)-1})$$

$$< b(d(x_{p(n)}, x_{q(n)-1})) \max \{d(x_{p(n)}, Tx_{p(n),} dx_{p(n)-1}, Tx_{q(n)-1})\}$$

$$+ c(d(x_{p(n)}, x_{q(n)-1})) \min \{d(x_{p(n)}, Tx_{q(n)-1}), d(x_{q(n)-1}), d(x_{q(n)-1}, Tx_{p(n)-1})\}$$

Case (C). $x_{p(n)+1} \in Q, x_{q(n)+1} \in P$.

$$d(x_{p(n)}, x_{q(n)+1}) = d(x_{q(n)-1}), Tx_{f}q(n))$$

$$\leq b(d(x_{p(n)-1}, x_{q(n)})) \max \{d(x_{p(n)}, x_{q(n),} dx_{q(n)-1}, Tx_{q(n)-1})\}$$

$$+ c(d(x_{p(n)}, x_{q(n)-1})) \min \{d(x_{p(n)}, Tx_{q(n)-1}, Tx_{p(n)-1})\}$$

Case (D). $x_{p(n)+1}, x_{q(n)+1} \in Q$.

$$d(x_{p(n)}, x_{q(n)}) = d(Tx_{p(n)-1}, Tx_{q(n)-1})$$

$$\leq b(d(x_{p(n)-1}, x_{q(n)-1})) \max \{d(x_{p(n)-1,}, x_{\hat{p}(n),} d(x_{q(n)-1}, x_{q(n)})\}$$

$$+ c(d(x_{p(n)_{-1}}, x_{q(n)-1})) \min \{d(x_{p(n)-1}\, x_{q(n)}), dx_{q(n)-1}, x_{p(n)})\}$$

Using the facts that

$$\lim d(x_{p(n)}, x_{q(n)-1}) = \lim d(x_{p(n)-1,} x_{q(n)}) = \lim d(x_{p(n)-1}, x_{q(n)-1}) = \varepsilon,$$

we have, for all n sufficiently large, upon adding the inequalities in Cases (a) - (d),

$$\begin{aligned} & d(x_{p(n)+1}, x_{q(n)+1}) + d(x_{p(n)+1}, x_{q(n)}) + d(x_{p(n)}, x_{q(n)+1)} + d(x_{p(n)}, x_{q(n)}) \\ & \quad \le b(\varepsilon) \max\{\tau_{p(n)}, \tau_{q(n)}\} + c(\varepsilon) \min\{d(x_{p(n)}, x_{q(n)+1}), d(x_{q(n)}, x_{p(n)+1})\\ & \quad + b(\varepsilon/2) \max\{\tau_{p(n)}, \tau_{q(n)}\} + c(\varepsilon/2) \min\{d(x_{p(n)}, x_{q(n)}), d(x_{q(n)-1}, x_{p(n)})\\ & \quad + b(\varepsilon/2) \max\{\tau_{p(n)-1}, \tau_{q(n)}\} + c(\varepsilon/2) \min\{d(x_{p(n)-1}, x_{q(n)+1}), d(x_{q(n)}, x_{p(n)})\\ & \quad + b(\varepsilon/2) \max\{\tau_{p(n)-1}, \tau_{q(n)-1}\} + c(\varepsilon/2) \min\{d(x_{p(n)-1}, x_{q(n)}), d(x_{q(n)-1}, x_{p(n)})\}\end{aligned}$$

Taking the limit as $n \to \infty$ gives

$$4\varepsilon \le c(\varepsilon)\varepsilon + c(\varepsilon/2)\, 3\varepsilon < 4\varepsilon,$$

a contradiction. Therefore $\{x_n\}$ is Cauchy, hence convergent to a point z.

We shall now show that z is a fixed point of T. Since $\tau_n > 0$ for all n, there exists a subsequence $\{x_{h(n)}\}$ of $\{x_n\}$ such that $x_{h(n)} \ne z$. In turn, there must be a subsequence of $\{x_{h(n)}\}$ consisting only of points of P or points of Q. Without loss of generality we shall assume two cases.

Case (E), $\{x_{h(n)}\} \subset Q$, and Case (F), $\{x_{h(n)}\} \subset P$.

For Case (E), $x_{h(n)} \in Q$ implies that $x_{h(n)+1} \in P$. Define $\rho_n := d(z, x_n)$.

$$d(x_{h(n)+1}, Tz) \le b(\rho_{h(n)}) \max\{\tau_{h(n)}, d(z, Tz)\} + c(\rho_{h(n)}) \min\{d(x_{h(n)}, Tz), \rho_{h(n)+1}\}.$$

Fix $\varepsilon > 0$. Then there exists an N such that $n > N$ implies that $\rho_{h(n)} < \varepsilon$.

Therefore, $c(\rho_{h(n)}) > c(\varepsilon)$, or $-c(\rho_{h(n)}) < -c(\varepsilon)$. Then

$$d(x_{h(n)+1}, T_z) \le (1 - c(\varepsilon)).\ \max\{\tau_{h(n)}, d(z, Tz)\} + \min\{d(x_{h(n)}, Tz), \rho_{h(n)+1}\}.$$

Taking the limit as $n \to \infty$ yields $d(z, Tz) \le (1 - c(\varepsilon))d(z, Tz)$, which implies that $z = Tz$.

For Case (F), we get

$$\begin{aligned} d(x'_{h(n)+1}, Tz) &= d(Tx_{h(n)}, Tz) \\ &\le b(d(x_{h(n)}, z)) \max\{d(x_{h(n)}, x'_{h(n)+1}), d(z, Tz)\} \\ &\qquad + c(\rho_{h(n)}, z)) \min\{d(x_{h(n)}, Tz), d(z, x'_{h(n)+1})\}.\end{aligned}$$

Fix $\varepsilon > 0$ and choose n large enough so that $\rho_{h(n)} < \varepsilon$. Then

$$(13)\quad \begin{aligned} d(x'_{h(n)+1}, Tz) &\le (1 - c(\varepsilon)) \max\{d(x_{h(n)}, x'_{h(n)+1}), d(z, Tz)\} \\ &\qquad + \min\{d(x_{h(n)}, Tz), d(z, x'_{h(n)+1})\}.\end{aligned}$$

From Proposition 1, $d(z, x'_{h(n)+1}) \leq \max\{(d, x_{h(n)}, d(z, x_{h(n)+1})\}$. Taking the limit of (13) as $n \to \infty$ yields

$d(z, Tz) \leq (1 - c(\varepsilon))\, d(z, Tz)$,

which implies that $z = Tz$.

The uniqueness of the fixed point follows from the Lemma of [2], since definition (2) is a special case of (24).

Theorem 2. *Let K be a nomempty compact subset of a Banach space $X, T : K \to X, T$ continuous and such that $x \in \partial K$ implies $Tx \in K$. Suppose that, for each $x, y \in K, x \neq y$,*

$$d(Tx, Ty)\ , b(x, y) \max\{d(x, Tx), d(y, Ty)\} + c(d(x, y) \min\{d(x, Ty), d(y, Tx)\},$$

where b and c are nonnegative decreasing functions from $R^+ \to [0, 1)$ such that $b(t) + c(t) \leq 1$ for all $t > 0$.Then T has a unique fixed point.

Proof. Let $\{x_n\}$ be a general orbit of x_0. By following the first part of the proof of Theorem 1, there are two changes. One is that $\leq$ is replaced every where by strict inequality. The other is that $c(\tau_n) \neq 1$. For, if $c(\tau_n) = 1$, then, from inequality (3), $b(\tau_n) = 0$, and one obtains the contradiction $d(x'_{n+1}, x_{n+2}) < d(x'_{n+1}, x_{n+2})$. In inequality (4) one obtains the inequality $\tau_{n+1} < \tau_{n-1}$.

Since X is compact $\{x_n\}$ has a convergent subsequence $\{x_{n(i)}\}$. Call the limit z.

Then $\{x_{n(i)}\}$ has one of the following three properties :

(P_1) $\{x_{n(i)}\}$ has a subsequence $\{x_{n(k)}\}$ such that $x_{n(k)+1}$ and $x_{n(k)+2} \in P$, or no such subsequence exists. Therefore, for all k sufficiently large, one must have either

(P_2) for all n sufficently large, the subsequence $\{x_{2n}\} \subset Q$, or

(P_3) for all n sufficiently large, the subsequence $\{x_{2n}\} \in P$.

If condition (P_1) is satisfied, then one obtains inequality (6). Taking the limit as $k \to \infty$ yields, using the continuity of T,

(14) $d(z, Tz) = d(Tz, T^2z) = d(z, Tz)$.

If condition (P_2) is satisfied, then $\{\tau_{2n}\}$ is monotone decreasing, hence convergent. If (8) is satisfied, (14) follows from (9) by taking the limit as $h \to \infty$.

If (7) is satisfied, then, by the triangular inequality,

$$\tau_{2n(h)} - d(x_{2n(h)-1}, x_{2n(h)+1}) \leq d(x_{2n(h)-1}, x_{2n(h)}) \leq d(x_{2n(h)-1}, x'_{2n(h)})$$
$$= d(Tx_{2n(h)-2}, Tx_{2n(h)-1}) \leq \tau_{2n(h)-2},$$

by Case (b). Taking the limit as $h \to \infty$ yields $\lim_h d(x_{2n(h)-1}, x_{2n(h)+1}) = 0$ and $\lim_h d(x_{2n(h)-1}, x'_{2n(h)}) = \lim_h \tau_{2n(h)}$, which implies (14).

The proof for (P_3) is similar.

Suppose that $z \neq Tz$. Then, from (14),

$$d(z, Tz) = d(Tz, T^2z) < b(z, Tz) \max \{d(z, Tz)\, d(Tz, T^2z)\} < d(z, Tz),$$

a contradiction. Therefore. $z = Tz$.

Uniqueness of the fixed point follows in Theorem 1.

Theorems 3.1 and 4.1 of [1] are special cases, respectively, of Theorems 1 and 2 of this paper.

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