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(Dedicated to Professor J.N. Kapur on his 70th Birthday)

A GENERALIZED STUDY OF A VISCOUS INCOMPRESSIBLE FLUID FLOW THROUGH VARIOUS CROSS SECTIONS OF A TUBE

By

N. Yadav

Department of Mathematics and Computer Science
National University of Lesotho
Lesotho (SOUTHERN AFRICA)

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ABSTRACT

An entirely new technique has been developed to study the velocity disrribution of viscous fluids flowing through tubes whose cross sections are any rectilinear figure of n-sides.

1. INTRODUCTION

The number of cases of viscous incompressible fluids flowing through channels of various cross section viz, rectangle, equilateral triabgle, right angle triangle, between parallel plates, elliptic section of tubes [1] have been discussed by various investigators. Besides these, the velocity distribution over the cross section of tube has also been determined for section bounded by (i) confocal ellipses (ii) a line and a parabola (iii) two confocal parabolas and part of their common axis (iv) two confocal parabolas etc. by different investigators [1]. In this paper an entirely new technique has been developed to study the velocity distribution when the cross section is any rectilinear figure of n sides.

2. PRELIMINARIES

In case of steady motion along a pipe of uniform cross section, if z-axis is taken parallel to the length of the pipe;

$$u = v = 0;$$

$$\frac{\partial w}{\partial z} = 0 \text{ and } \frac{\partial w}{\partial t} = 0.$$

So Navier Stokes equations reduce to a single equation

$$\textbf{(2.1)} \quad \rho z - \frac{\partial p}{\partial z} + \mu \nabla^2 w = 0.$$

Assuming z to be zero and the pressure gradient $\frac{\partial p}{\partial z}$ to be constant $(= -h)$, the equation (2.1) reduces to,

$$\textbf{(2.2)} \quad \nabla^2 w = -\frac{h}{\mu} = -K \text{ (say)}.$$

Hence, finally we get,

(2.3) $$\frac{\nabla^2 w}{\partial x^2} + \frac{\nabla^2 w}{\partial y^2} = -K$$

which has a particular solution

(2.4) $$w = -\frac{K}{2}\frac{Ax^2 + By^2}{A+B}.$$

If $\Phi(x, y)$ is also a solution of $\nabla^2 \varphi(x, y) = 0$. Then (2.3) has a general solution

(2.5) $$w = \varphi(x, y) - \frac{K}{2}\frac{Ax^2 + By^2}{A+B}.$$

When the liquid touches the wall of the pipe, $w = 0$ and we have

(2.6) $$\varphi(x, y) - \frac{K}{2}\frac{Ax^2 + By^2}{A+B} = 0$$

which gives the boundry of the cross section of the pipe for flow through which **(2.5)** holds.

$\varphi(x, y)$ is circular harmonics which are known to be of one of the froms of the harmonics $r^n \cos(n\theta)$, $r^n \sin(n\theta)$ $\frac{\cos(n\theta)}{r^n}$, $\frac{\sin(n\theta)}{r^n}$, $\log r$ and θ or their combinations.

In cartesian coordinates the harmonics are given by

$x^n - {}^nc_2\, x^{n-2}y^2 + {}^nc_4 x^{n-4}y^4 \ldots$ and ${}^nc_1 x^{n-1}y - {}^nc_3 x^{n-3}y^3 \ldots,$

which correspond to $r^n\cos(n\theta)$ and $r^n\sin(n\theta)$. Dividing them by $(x^2 + y^2)^n$, we get the forms corresponding to $\frac{\cos(n\theta)}{r^n}$ and $\frac{\sin(n\theta)}{r^n}$.

The harmonics of the type $\log r$ and θ in polar coordinates are given by $\log(x^2 + y^2)$ and $\tan^{-1}(\frac{y}{x})$ in cartesian coordinates.

These harmonics result from the complex transformation,

$$w = z^n \text{ and } w = \log z,$$

where n is positive or negative integer. Unlimited number of harmonics are derived from the transformation $w = f(z)$.

First case of interest occurs when $\varphi(x, y)$ is at most of second degree in (x, y). In this case

(2.7) $$w = a_2(x^2 - y^2) + 2b_2xy + a_1x + b_1y + a_0 - \frac{K}{2}\frac{Ax^2 + By^2}{A+B}.$$

The boundary is given by

(2.8) $$\left[a_2 - \frac{KA}{2(A+B)}\right]x^2 + 2b_2xy - \left[a_2 + \frac{KB}{2(A+B)}\right]y^2 + a_1x + b_1y + a_0 = 0$$

which is a conic section.

If equation (**2.8**) is written as,

(**2.9**) $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, then

(**2.10**) $w = ax^2 + 2hxy + 2gx + 2fy + by^2 + c$

gives the velocity function admissible. The point of maximum velocity coincides with the centre because they are given by the same set of equations.

If (x', y') is the centre, then the velocity at centre is,

(**2.11**) $w_c = gx' + fy' + c$.

Constants a, b and K are related by the equations;

$$\nabla^2 w = 2a + 2b = -K,\ a + b = -\frac{K}{2}.$$

For special values of the constants the general form (2.10) assumes one of the following forms :

(**2.12**) $w = -\dfrac{K}{2}(x^2 + y^2 - \alpha^2)$

when the cross section is the circle, $x^2 + y^2 = \alpha^2$

(**2.13**) $w = -\dfrac{K}{2}\dfrac{a^2b^2}{a^2 + b^2}\left[\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1\right]$

when the cross section is ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(**2.14**) $w = \dfrac{K}{2}\dfrac{a^2b^2}{b^2 - a^2}\left[\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1\right]$

when the cross sectiion is hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is not a closed curve therefore channels must be constructed instead of pipes.

Flow in parabolic channels may be studied by the form

(**2.15**) $w = -\dfrac{K}{2}(x^2 - 4y)$

while the flow between two parallel planes may be formulated in the forms

(**2.16**) $w = \dfrac{K}{2}(x^2 - a^2)$

(2.17) $w = \frac{K}{2}(x^2 - mxy)$ or more symmetrical form

(2.18) $w = \frac{k}{2(1-m^2)}(x^2 - m^2y^2)$

gives the flow in V-shaped channels of two intersecting planes.

These special forms and many others have been studied in detail by many authors. Harmonics of the type cosh (nx) cos (ny) in series with suitable coefficients have been employed to form velocity fuction for flow when the cross section is rectangle $x = \pm a, y = \pm b$, and also when it is right angled triangle. Third degree harmonics have been employed for equilateral triangle. But the case of any triangle or any parallelogram or other rectilinear figure have not been solved as yet.

In what follows, we propose to develope an entirely new technique and find out the velocity distribution when the cross section is any rectilinear figure of n-sides. We shall limit our selves to quadratic form of the velocity function and shall not use higher degree harmonics nor harmonics like cosh (nx) cosh (ny).

3. BOUNDARY OF THE RECTILINEAR FIGURE OF n-SIDES

The polar equation of straight line is

(3.1) $\rho \cos(\theta - \alpha) = p$.

If there are n lines, their equation may be written as,

(3.2) $\rho \cos(\theta - \alpha_r) = p, \qquad (r = 1, 2, 3, \ldots, n)$

If β_r is the angular coordinate of the point of intersection of r^{th} and $(r+1)^{th}$ lines then β_r satisfies,

(3.3) $\cos(\beta_r - \alpha_r)p_{r+1} = \cos(\beta_r - \alpha_{r+1})p_r$.

The r^{th} line extends from

$\theta = \beta_{r-1}$ to $\theta = \beta_r$.

Since the figure is closed, $\beta_0 = \beta_n$

The figure under consideration may be said to have a boundary given by,

(3.4) $\rho \cos(\theta - \alpha_r) = p_r \,;\, \beta_{r-1} < \theta < \beta_r$.

This boundary may be represented more elegantly by means of functions which we shall call $\phi(\theta, \beta\beta'); \beta < \beta'$

$\phi(\theta, \beta\beta')$, have the following definition :

$\Phi(\theta, \beta\beta') = 1$ for $\beta < \theta < \beta'$

$\phi(\theta, \beta\beta') = 0$ for $\beta' < \theta < 2\pi + \beta$.

Hence the whole boundary is given by,

(3.5) $$\sum_{r=1}^{r=n} [\rho \cos(\theta - \alpha_r) - p_r]\, \phi(\theta, \beta_{r-1}\beta_r) = 0.$$

The same boundary is also given by,

(3.6) $$\sum_{r=1}^{r=n} \lambda_r [\rho \cos(\theta - \alpha_r) - p_r]\, \phi(\theta, \beta_{r-1}\, \beta_r) = 0$$

where λ_r are arbitrary constants.

In what follows a quadratic form is preferable for the boundary, so in place of **(3.5)** or **(3.6)** we shall denote the boundary by,

(3.7) $$\sum_{r=1}^{r=n} [p_r^2 - \rho^2 \cos^2(\theta - \alpha_r]\, \phi(\theta, \beta_{r-1}\beta_r) = 0$$

(3.8) $$\sum_{r=1}^{r=n} \lambda_r [p_r^2 - \rho^2 \cos^2(\theta - \alpha_r)]\, \phi(\theta, \beta_{r-1}\, \beta_r) = 0$$

for $\beta_{r-1} < \theta < \beta_r$ in each case.

In fact $p^2 = \rho^2 \cos^2(\theta - \alpha)$, represents pair of lines both being parallel and on opposit sides of the origin at distance p. Therefore (3.7) or (3.8) represent two equal rectilinear figures each of which is obtainable by rotating the other round the origin through 180^0. This fact will have to be taken into account in calculating total flow per unit time.

It should be noted that functions like $\phi(\theta, \beta\beta')$ are always possible to construct and these may be analytically expressed for $0 < \theta < 2\pi$ as,

$$\Phi(\theta, \beta\beta') = \sum_{s=0}^{s=\infty} C_s \cos(s\theta)$$

where C_s are the Fourier coefficients calculated in the usual manner **[2]**.

The values of $\phi(\theta, \beta\beta')$ relevant for this calculation are :

$$\phi = 0 \text{ for } 0 < \theta < \beta$$
$$\phi = 1 \text{ for } \beta < \theta < \beta'$$
$$\phi = 0 \text{ for } \beta' < \theta < 2\pi$$

Therefore

$$C_s = \frac{2}{\pi} \int_0^{\pi} \phi(\theta, \beta\beta') \cos(s\theta)\, d\theta$$

$$= \frac{2}{\pi} \int_{\beta}^{\beta'} \cos(s\theta)\, d\theta$$

$$= \frac{2}{s\pi} (\sin s\beta' - \sin s\beta) \text{ and}$$

$$C_0 = \frac{1}{\pi} \int_{\beta}^{\beta'} d\theta = \frac{\beta' - \beta}{\pi}.$$

Thus equation (3.8) written in full is,

(3.9) $$\sum_{r=1}^{r=n} \lambda_r \left[p_r^2 - \rho^2 \cos^2(\theta - \alpha_r)\right] \left[\frac{\beta_r - \beta_{r-1}}{\pi} + \sum_{s=1}^{s=\infty} \frac{2(\sin s\beta_r - \sin s\beta_{r-1})}{s\pi}\right].$$

4. DETERMINATION OF THE VELOCITY DISTRIBUTION

Here we shall be studying the velocity distribution given by,

(4.1) $$w = \sum_{r=1}^{r=n} \frac{K}{2} \left[p_r^2 - \rho^2 \cos^2(\theta - \alpha_r)\right] \phi(\theta, \beta_{r-1}\, \beta_r)$$

For $\beta_{r-1} < \theta < \beta_r$

$$w = \frac{K}{2} \left[p_r^2 - \rho^2 \cos^2(\theta - \alpha_r)\right]$$

$$w = \frac{K}{2} \left[p_r^2 - x^2 \cos^2\alpha_r - y^2 \sin^2 \alpha_r - 2xy \cos \alpha_r \sin \alpha_r\right]$$

so that,

$$w = \frac{K}{2} \left[- 2 \cos^2 \alpha_r - 2 \sin^2 \alpha_r\right] = - K.$$

Also $w = 0$ gives the boundary considered in 3 i.e.

$$\sum_{r=1}^{r=n} \left[p_r^2 - \rho^2 \cos^2(\theta - \alpha_r)\right] \phi(\theta, \beta_{r-1}\, \beta_r) = 0.$$

Thus equations (4.1) give the form of velocity function for steady flow of an incompressible viscous fluid in pipe whose cross section is rectilinear figure of n sides.

So far we have taken p_r to be quite arbitrary but continuity considerations impose limitations on the value of p_r which we now proceed to dicuss.

The velocity function should be continuous from both the sides of the line $\theta = \beta_r$, so,

(4.2) $$\frac{K}{2} \left[p_r^2 - \rho^2 \cos^2(\beta_r - \alpha_r)\right] = \frac{K}{2} \left[p_{r+1}^2 - \rho^2 \cos^2(\beta_r - \alpha_{r+1})\right]$$

for all values of ρ.

This is possible only if

$p_r = p_{r+1}$ and $\beta_r - \alpha_r = \alpha_{r+1} - \beta_r$. So we get,

(4.3) $p_1 = p_2 = p_3 = \ldots = p_n$ and every

(4.3) $p_1 = p_2 = p_3 = \ldots = p_n$ and every

(4.4) $\beta_r = \dfrac{\alpha_r + \alpha_{r+1}}{2}$

In fact two conditions **(4.3)** and **(4.4)** are identical, when one is satisfied, the other will also be satisfied. Under these conditions w takes the form,

(4.5) $$w = \sum_{r=1}^{r=n} \frac{K}{2}\left[p^2 - \rho^2 \cos^2(\theta - \alpha_r)\right]\left(\theta, \frac{\alpha_{r-1} + \alpha_r}{2} \frac{\alpha_r + \alpha_{r+1}}{2}\right)$$

or more simply for

$$\frac{\alpha_{r-1} + \alpha_r}{2} < \theta < \frac{\alpha_r + \alpha_{r+1}}{2}$$

(4.6) $$w = \frac{K}{2}\left[p^2 - \rho^2 \cos^2(\theta - \alpha_r)\right].$$

The boundary of the cross secticn of the pipe being formed of lines,

(4.7) $\rho \cos(\theta - \alpha_r) = \pm p;\ r = 1, 2, 3, \ldots n.$

The rectilinear figure given by **(4.7)** has the following properties :

(1) A circle can be inscribed in the figure touching all the sides.

(2) Origin or $\rho = 0$, is the centre of the circle.

(3) p is the radius of the in-circle. So that all the boundary lines are equidistant from the origin and all such figures can be described round a circle.

If L is the perimeter of the rectilinear figure then the perimeter of a similar figure indside with sides at distance of ξ from the outer side is,

$$\frac{L(p - \xi)}{p}.$$

So that the whole area of cross section is given by,

(4.8) $$A = \int_0^p \frac{L(p - \xi)}{p}\, d\xi = \frac{Lp}{2}$$

and total flow per unit time,

$$Q = \int_0^p \frac{K}{2}\left[p^2 - (p - \xi)^2\right]\frac{L(p - \xi)}{p}\, d\xi = \frac{KLp^3}{8}.$$

This must be the flow in pair represented by **(3.7)** or **(3.8)** therefore for a single pipe we get,

(4.9) $$Q = \frac{Klp^3}{16}.$$

Mean velocity

(4.10) $$U = \frac{Q}{A} = \frac{KLp^3}{16}\frac{2}{Lp} = \frac{Kp^2}{8}.$$

Maximum velocity

(4.11) $$c = \frac{Kp^2}{2}$$

(4.12) $$k = \frac{U}{KA} = \frac{Kp^2}{8}\frac{2}{KLp} = \frac{p}{4L}$$

(4.13) $$k' = \frac{c}{KA} = \frac{2Kp^2}{2KLp} = \frac{p}{L}.$$

5. SOME PARTICULAR CASES OF VISCOUS FLOW THROUGH TRIANGULAR AND RECTANGULAR CROSS SECTIONS

Triangles are figures with smallest numbers of sides that can be described round a circle.

The general formula for total flow per unit time for rectilinear figures in section **4** has been obtained as,

$$Q = \frac{KLp^3}{16}$$

If S is the area and s is the perimeter then

$$p = \frac{S}{s} \text{ and } L = 2s$$

Therefore,

(5.1) $$Q = \frac{2Ks}{16}\frac{S^3}{s} = \frac{KS^3}{8s^2}$$

It will be interesting to compare this result (5.1) with two known solutions for triangles given by Boussinesq. Boussinesq formulae for equilateral triangle given by,

$y = \pm\sqrt{3}x,\ y = b$, gives,

(5.2) $$Q = \frac{Kb^4}{60\sqrt{3}}$$

Here, in this case,

$$S = \frac{b^2}{\sqrt{3}},\ s - \sqrt{3}b\ .$$

Therefore from **(5.1)** we get

(5.3) $$Q = \frac{K}{8}\frac{b^6}{3\sqrt{3}}\frac{1}{3b^2} = \frac{Kb^4}{72\sqrt{3}}.$$

Hence, **(5.2)** and **(5.3)** give values of Q in the ratio of 6:5.

Solution obtained by Proudman and others for rightangled triangle given by,

$x = a, y = a, x + y = 0$, gives

$$(5.4) \qquad Q = \frac{K}{4a}\left[\frac{4}{3}a^5 - \sum_{n=0}^{\infty} N^{-5} \coth (Na)\right]$$

where $2Na = (2n + 1)\pi$.

In this case

$S = \frac{a^2}{2}$; $s = (1 + 1/\sqrt{2})a$.

Therefore, Q from **(5.1)** is given by,

$$(5.5) \qquad Q = \frac{K\,2a^6}{64(3 + 2\sqrt{2})a^2} = \frac{Ka^4}{186.496}\,.$$

Thus formula **(5.1)** for Q in case of triangular cross section does not agree with the two known solutions. But when the number of sides is made indefinitely great and the rectilinear figure becomes a circle of radius a ;

$S = \pi a^2$ and $s = \pi a$

then from **(5.1)** we get,

$$(5.6) \qquad Q = \frac{K\pi^3 a^6}{8\pi^2 a^2} = \frac{\pi k a^4}{8},$$

which agrees with the known solution for pipe of circular cross section. This result **(5.6)** may be taken as a clue to the correctness of the solution obtained in previous section. However, the result can be experimentally checked for the correctness of the formulae of previous section.

In case of a square where sides are $2a$;

$S = 4a^2, s = 4a;$

Q is given by

$$Q = \frac{64\,K\,a^6}{128\,a^2} = \frac{K\,a^4}{2}$$

and $\qquad k = \frac{1}{32}, k' = \frac{1}{8}\,.$

In similar way total flow for other cases of rectangle may be calculated. The result of this section can be easily extended to the cases when the cross section of the pipe is made up of arc of conic sections instead of straight lines.

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