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(Dedicated to Professor S.A.Naimpally on his 62th Birthday)

NATURE OF EQUILIBRIUM POINTS IN CERTAIN SYMMETRIC MATRICES)

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ABSTRACT

In this paper we apply the fixed point algorithm to investigate th nature of the quilibrium points in the symmetric matrices that occur in convex quadratic programming and in many decision problems. Lmak, Cottle and Dantzig have already shown that every linear programming problem and convex quadratic programming problem can be transformed into a linear complementary method based on fixed point algorithm has been applied hr, and we also discuss the behaviour of equilibrium points for symmetric matrix M of order 3 statisfying the following form of LCP.

$$I_w - M_z - z_0 e = q$$

$$w \geq 0 \ ; z \geq 0 \ ; z_0 \geq 0$$

$$w_i z_i = 0 \ ; i = 1, 2, ..., n$$

1. Introduction

To solve the non-linear mathematical problems there is no unique simplex type procedure as is in the cases of linear mathematical problems. Therefore, many non-linear algorithms, known as quadratic programming, separable programming, geometric programming, fractional programming, and dynamic programming, have been developeeed according to the nature of the non-linear problem. All these important algorithms that are used to find the solution of non-linear problems aree based on the Khun-Tucker condition. Due to complexity of these algorithms, Lemke [3] and Cottle and Dantzig [1], have shown that every programming problem and convex quadratic problem can be transformed into linear complementary problem. Thus the LCP provides a unifying format for studying a large class of important problems in mathematical programming and in there theory of matrix games. Lemke and Howson [2] developed a complementary pivot method for solving such LCP pertaining to the problem of computing equilibrium points in bimatrix gaem. Prior, to the Lemke methods, Cottle and Dantzig [1] proposed an algorithm known as principle pivoting method to solve *LCP*, but Lemke's algorithm is logically simpler than that of Cottle and Dantzig. Moreover, it can be used to process the wider class of complementary problem. Lemke [3] has also extended his method for solving a general *LCP*. This general method is normally known as complementary pivot method or Lemke's complementary pivot method.

The main characteristics of Lemke's technique for the solution of general form of *LCP* given by

(1.1) $$Mz - w = q\ ;\ z \geq 0\ ; w \geq 0,$$

where M is any square matrix

It is based on the concept of non-degeneracy and the generation of adjacent extreme point path which terminates in an equilibrium point, if it exists. Geometrically, the system of the form (1.1) can be visualised as convex polyhedron in z-space of points satisfying :

$$Mz \geq q\ ;\ z \geq 0$$

The path of the points generated consist those points for which

$$z^T w = \sum_{i=1}^{n} z_i w_i$$

that is, points for which the sum has at most summand $z_i w_i$ positive for fixed i.

Moreover, a point of (1.1) which reduces $z^T w = 0$ is called an equilibrium point.

In this paper we have considered a complementary problem of the form :

$$Iw - Mz - z_0 e = q$$

$$w \geq 0\ ; z \geq 0\ ; z_0 \geq 0,$$

where M is a symmetric 3×3 matrix whose equilibrium points exist.

We have tried to show under what conditions the equilibrium points for symmetric matrix remains unaltered, and also that any new equilibrium point is a multiple of the previously obtained equilibrium point.

2. Notations and Definitions

The General form of the set of LCP in R^n is denoted by :

(A) $$Z = \{ z : w - Mz = q : z \geq 0\ ; w \geq 0 \}$$

2.1 Definition :

In any *LCP* with M as the data, the pair of column vectors $\{ I.j . -M.j \}$ is called as j^{th}. Complementary pair of column vector for

$j = 1,2,\ldots,n$ denoted by $A._j$

The ordered set of vectors $(A._1 \ldots A._n)$ is called as complementary set of vectors.

To determine these complementary set of vectors $(A_{.1} \ldots A_{.n})$; for any given pair (q,m) in *LCP*, the process similar to *LPP* is applied.

3. Preliminary The following result has been proved by Lemke [3] to show the existence and characterisation of equilibrium points.

3.1 Theorem : For fixed s, if Z is non-degenerate, Z_s is either empty or the union of disjoint adjacency paths of Z. Then the set S of equilibrium points of Z is precisely the set of end-points of adjacency paths comprising Z_s.

Based on theorem (3.1) the following lemma and theorem gives the existence of equilibrium points in *LCP*.

3.2 Lemma : Z is empty if there exists a $u > 0$ satisfying :

(3.2.1) $$M^T u \le 0 \text{ ; and } u^T q > 0.$$

3.3 Theorem : Let Z be non-degenerate and have the property that for some s, Z_s contains precisely one ray of Z. The Z has an odd number of equilibrium points.

The consequence of the above theorem is the following corollary for the bimatrix game.

3.4 Corollary :Let Z be non-degenerate if $q = e$, and $M > 0$ then Z has an odd number of equilibrium points.

The following theorm gives the uniqueness condition of equilibrium points.

3.5 Theorem :If Z is non-degenerate and M satisfies $z^T Mz \ge 0$ (that is, if M is non- negative definite) then there exists at most one equilibrium point.

By similar process the equilibrium points for more wider class of problems of the form :

(B) $$Z^* = \{ z^* : M_Z + z_0 e - w = q : w, z \ge 0 , z_0 \ge 0 \} \qquad R^{n+1}$$

can be obtained

where $z^* = \begin{pmatrix} z \\ z_0 \end{pmatrix}$. the details can be found in [3].

Since in this paper we mainly concentrated on the computation procedure applied to *LCP* for the determination of equilibrium points when matrix M is symmetric, so the steps of alogrithm that has been used to compute the equilibrium points *LCP* are similar to *LPP* problem except the following :

(i) The *LCP* is taken in the form as :

$$Iw - Mz - z_0 e = q$$

$$; w \ge 0,\ z \ge 0, z_0 \ge 0$$

expressed in tableau form as

w	z	z_0	
I	$-M$	$-e$	q

(a) If at some stage the variable z_0 becomes the outgoing variable. At this stage the basic vector at the end if the pivot step is a complementary feasible basic vector for the LCP given by (i) and thus the basic solution corresponding to it is a solution of the LCP (i).

(b) If at some stage z_0 may still be a basic vector and the incoming variable just the repetition of the previous steps or the pivot column may be non-positive. When this happen the algorighm terminate as it has reached the stage where, it is unable to solve the LCP given by (i). This type of termination is called ray termination.

4. Main Result. To study the behaviour of symmetric matrix in LCP while determining the equilibrium points, let us first consider following symmetric matrix in LCP given below :

Let

4.1 $$Iw - Mz - z_0 e = q$$

where

4.2 $$M = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

and

4.3 $$q = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$$

On applying (4.2) and (4.3) in (4.1) the initial tableau form takes the form as :

(4.4)

w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
1	0	0	– 4	– 2	– 1	– 1	– 4
0	1	0	– 2	– 4	– 2	(– 1)	– 6
0	0	1	– 1	– 2	0	– 1	– 2

As per LCP algorithm the starting pivot point is – 1 of second element in z_0 corresponding to least value – 6 of q. On applying LCP rules so that Z_0 to be the element of basic vector as well as to obtain equilibrium points, the integration step are shown in three subsequents tables as given below :

4.5

Basic variable	w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
w_1	1	−1	0	−2	②	1	0	2
z_0	0	−1	0	2	4	2	1	6
w_3	0	−1	1	1	2	2	0	4
z_2	1/2	−1/2	0	−1	1	1/2	0	1
z_0	−2	1	0	⑥	0	0	1	2
w_3	−1	0	1	3	0	1	0	2
z_2	1/6	−1/3	0	0	1	1/2	1/6	4/3
z_1	−1/3	1/6	0	1	0	0	1/6	1/3
w_3	0	−1/2	1	0	0	1	−1/2	1

Thus we have z = (1/3, 4/3, 0) and w = (0,0,1), satisfying $(w)_i, (z)_i = 0 (w), (z), = 0$. Hence

(4.6) $$z = (1/3, 4/3, 0)$$

is an equilibrium point.

Again, on replacing the third element of first row of symmetric matrix of (4.2) by another element so that M remains a symmetric matrix, we observe that equilibrium points remains unaltered except value of w changes which can be seen by considering the collowing matrix and the corresponding tableau as shown below :

Let

(4.7) $$M = \begin{pmatrix} 4 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 0 \end{pmatrix} \quad \text{and} \quad (4.8)\ q = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$$

(4.9)

w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
1	0	0	−4	−2	−3	−1	−4
0	1	0	−2	−4	−2	(−1)	−6
0	0	1	−3	−2	0	−1	−2

By applying integration steps similar to (4.5) we have z = (1/3, 4/3, 0) but

$$w = (0,0,5/3), \text{ yet satisfying } (w)_i (z)_i = 0$$

thus

(4.10) $z = (1/3, 4/3, 0)$ ·

is again an equilibrium point.

But if we replace third element of the second row of the matrix (4.2) some other element then the equilibrium point does not exist as shown below :

Let

(4.11) $$M = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 5 \\ 1 & 5 & 0 \end{pmatrix}$$

and

(4.12) $$q = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$$

we have

(4.13)

w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
1	0	0	− 4	− 2	− 1	− 1	− 4
0	1	0	− 2	− 4	− 5	(− 1)	− 6
0	0	1	− 1	− 5	0	− 1	− 2

(4.14)

Basic variables	w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
w_1	1	− 1	0	− 2	2	④	0	2
z_0	0	− 1	0	2	4	5	1	6
w_3	0	− 1	1	1	− 1	5	0	4
z_3	1/4	− 1/4	0	− 1/2	1/2	1	0	1/2
z_0	− 5/4	1/4	0	9/2	3/2	0	1	7/2
w_3	− 5/4	1/4	1	(7/2)	− 7/2	0	0	3/2
z_3	− 3/8	− 3/4	2/7	0	0	1	0	10/14
z_0	5/14	− 1/14	− 9/7	0	− 3	0	1	11/7
z_1	− 5/14	1/14	(2/7)	1	− 1	0	0	3/7
z_3	− 1/56	23/28	0	− 1	1	1	0	2/7
z_0	− 38/28	1/49	0	9/2	− 15/2	0	1	49/14
z_3	− 5/4	1/4	1	7/2	− 7/2	0	0	3/2

(Since there is a repetition of the tableau, so it has no equilibrium point.

Moreover, we have also observed that if we alter any one element of the diagonal then a new equilibrium point is obtained. Besides, if we consider the element of q in (4.3) as its some multiple, then correspondingly equilibrium points also increase by the same multiple.

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