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INEQUALITIES INVOLVING *H*-FUNCTIONS

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ABSTRACT

Six inequalities involving Fox's *H*-functions are established here with the help of certain known inequalities involving ${}_2F_1$ and ${}_3F_2$. Four inequalities involving Fox's *H*-function and ${}_2F_1$ are obtained as particular cases of the results established.

1. INTRODUCTION

Koti [2] established a number of inequalities involving the hypergeometric functions ${}_2F_1$ and ${}_3F_2$ with the help of certain statistical techniques. The aim of this paper is to establish six inequalities involving Fox's H-function by employing thee inequalites established by Koti [2] and following the approach adopted in Raina and Koul [3]. As a particular cases of the inequalities established here, four inequalites involving Fox's *H*-functions and ${}_2F_1$ are obtained.

The Fox's *H*-function is defined and represented in the following manner :

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_L \theta(s)\,x^s\,ds\,,\ \omega=(-1)^{1/2} \qquad \text{... (1.1)}$$

where

$$\theta(s)=\frac{\prod\limits_{j=1}^{m}\Gamma(b_j-B_js)\prod\limits_{j=1}^{n}\Gamma(1-a_j+\alpha_js)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j+\beta_js)\prod\limits_{j=n+1}^{p}\Gamma(a_j-\alpha_js)}\,.$$

For further details, existence and convergence conditions of $H^{m,n}_{p,q}[\,.\,]$, we refer to Srivastava et al. [4, *pp*.10-13].

2. RESULTS REQUIRED

(a) The following inequalities due to Koti [2,*pp*.391-395, theorem (2.1) and equations (3.5), (4.5)] ar rquired in the sequel :

Let k, m, n and λ be positive intgers and let k' be an odd positive integer. Then for $\alpha, \beta > 0$ and $0 < x < 1$:

$$\sum_{j=0}^{k} \binom{2k}{rj} \frac{(1-x)^{-2j}}{(\beta+2k-2j)} {}_2F_1\left(\begin{matrix}\beta+2k-2j, & 1-\alpha-2j\,; \\ 1+\beta+2k-2j & ;\end{matrix}\; x\right)$$

$$\geq \sum_{j=0}^{k-1} \binom{2k}{2j+1} \frac{(1-x)^{-2j-1}}{(\beta+2k-2j-1} {}_2F_1\left(\begin{matrix}\beta+2k-2j-1, -\alpha-2j\,; \\ \beta+2k-2j \,;\end{matrix}\; x\right); \quad \text{... (2.1)}$$

$$\sum_{j=0}^{[k'/2]} \binom{2k}{2j+1} \frac{1(1-x)^{-2j-1}}{(\beta+k'-2j-2j-1)} {}_2F_1\left(\begin{matrix}\beta+2k-1, -\alpha-2j\,; \\ \beta+k'-2j \,;\end{matrix}\; x\right)$$

$$\geq \sum_{j=0}^{[k'/2]} \binom{k'}{2j} \frac{(1-x)^{-2j}}{(\beta+k'-2j)} {}_2F_1\left(\begin{matrix}\beta+k'-2j, & 1-\alpha-2j\,; \\ 1+\beta+k'-2j & ;\end{matrix}\; x\right); \quad \text{... (2.2)}$$

$$\sum_{j=1}^{k} \frac{(-2k)_{2j}}{(2j)\,!} \frac{(1-x)^{2j}}{(\beta+2j)} {}_2F_1\left(\begin{matrix}\beta+2j, & 1-\alpha-m+2j\,; \\ 1+\beta+2j & ;\end{matrix}\; x\right)$$

$$\geq \sum_{j=0}^{k-1} \frac{(-2k)_{2j+1}}{(2j+1!} \frac{(1-x)^{2j+1}}{(\beta+2j+1)} 2F_1\left(\begin{matrix}\beta+2j\,, 2-2-m+2j \\ 2+\beta+2j \;\; ; x\end{matrix}\right); \quad \text{... (2.3)}$$

$$\sum_{j=1}^{[k'/2]} \frac{(-k')_{2j}}{(2j)\,!} \frac{(1-x)^{2j}}{(\beta+2j)} {}_2F_1\left(\begin{matrix}\beta+2j, & 1-\alpha-m+2j\,; \\ 1+\beta+2j & ;\end{matrix}\; x\right)$$

$$\geq \sum_{J=0}^{[k\,1/2]} \frac{(-k')_{2j+1}}{(2j+1)\,!} \frac{(1-x)^{2j+1}}{(\beta+2j+1)} {}_2F_1\left(\begin{matrix}\beta+2j+1, & 2-\alpha-m+2j\,; \\ 2+\beta+2j & ;\end{matrix}\; x\right) \quad \text{... (2.4)}$$

$$\sum_{j=0}^{k} \binom{2k}{2j} \frac{x^{-2j-1}}{(\gamma-n-2j)_\lambda} {}_3F_2\left(\begin{matrix}-n, \;\gamma-\alpha, \;\; \gamma-2j-n\,; \\ \gamma-2k-n, \gamma+\lambda-2j-n\,;\end{matrix}\; x\right)$$

$$\geq \sum_{j=0}^{k-1} \binom{2k}{2j+1} \frac{x^{-2j-2}}{(\gamma-n-2j-1)_\lambda} {}_3F_2\left(\begin{matrix}-n, \;\gamma-\alpha, \;\; \gamma-2j-1-n\,; \\ \gamma-2k-n, \gamma+\lambda-2j-1-n\,;\end{matrix}\; x\right)$$

$$\gamma > \alpha > n+2k\,; \quad \text{... (2.5)}$$

$$\sum_{j=0}^{[k'/2]} \binom{k'}{2j+1} \frac{x^{-2j-2}}{(\gamma-n-2j-1)_\lambda} {}_3F_2\left(\begin{matrix}-n, \;\gamma-\alpha, \;\; \gamma-2j-1-n\,; \\ \gamma-k'-n, \gamma+\lambda-2j-1-n\,;\end{matrix}\; x\right)$$

$$\geq \sum_{j=0}^{[k'/2]} \binom{k'}{2j} \frac{x^{-2j-1}}{(\gamma-n-2j)_\lambda}\ {}_3F_2\left(\begin{matrix} -n, & \gamma-\alpha, & \gamma-2j-n\ ; \\ \gamma-k'-n, & \gamma+\lambda-2j-n & ; \end{matrix}\ x \right),$$

$\gamma > \alpha > n + k'$. ... (2.6)

The rsults in (2.5) and (2.6) are obtained from Koti [2,*p*.395, *eqn*. (4.5)] by choosing α_2, β_2 suitably.

(b) The following integrals are required here :

Gupta and Olkha [1,p.207, eqs. (2.1), (2.2)] :

For $\mu, \nu > 0$, $Re(\rho) > 0$

$$\int_0^1 x^{\rho-1}\ (1-x)^{\alpha-\rho-1}\ {}_2F_1\left(\begin{matrix} \alpha, \beta\ ; \\ \gamma\ ; \end{matrix}\ x \right) \left[1 + z\left(\frac{x}{1-x} \right)^{\mu} \right]^{-\nu} dx$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\nu)\,\Gamma(\alpha)\,\Gamma(\gamma-\beta)}\ H_{3,3}^{3,2}\left[z \left| \begin{matrix} (1-\rho, \mu), (1-\nu, 1), (\gamma-\rho, \mu) \\ (0, 1), (\alpha-\rho, \mu), (\gamma-\beta-\rho, \mu) \end{matrix} \right. \right],$$

$Re(\alpha + \mu\nu) > \mathrm{Re}(\rho)$; ... (2.7)

$$\int_0^1 x^{\rho-1}\,(1-x)^{\sigma-1}\ {}_3F_2\left(\begin{matrix} -n, c_2, c_3\ ; \\ d_1, d_2\ ; \end{matrix}\ x \right) \left(1 + zx^{\mu} \right)^{-\nu} dx$$

$$= \frac{\Gamma(\sigma)}{\Gamma(\nu)} \sum_{r=0}^{n} \frac{(-n)_r\,(c_2)_r\,(c_3)_r}{(d_1)_r\,(d_2)_r\ r\,!}\ H_{2,2}^{1,2}\left[z \left| \begin{matrix} (1-\rho-r, \mu), (1-\nu, 1) \\ (0, 1), (1-\rho-\sigma-r, \mu) \end{matrix} \right. \right]$$

$Re\ (\sigma) > 0$. ... (2.8)

3. THE MAIN INEQUALITIES

For $\alpha, \beta, \mu, \nu, \sigma, \rho, z > 0$; k, m, n and λ being positive integers and k' an odd positive integer, we have :

$$\sum_{j=0}^{k} \binom{2k}{2j} H_{3,3}^{3,2}\left[z \left| \begin{matrix} (1-\rho, \mu), (1-\nu, 1), (1+\beta+2k-2j-\rho, \mu \\ (0, 1), (\beta+2k-2j-\rho, \mu), (\alpha+\beta+2k-\rho, \mu) \end{matrix} \right. \right]$$

$$\geq \sum_{j=0}^{k-1} \binom{2k}{2j+1} H_{3,3}^{3,2}\left[z \left| \begin{matrix} (1-\rho, \mu), (1-\nu, 1), (\beta+2k-2j-\rho, \mu) \\ (0, 1), (\beta+2k-2j-\rho-1, \mu), (\alpha+\beta+2k-\rho, \mu) \end{matrix} \right. \right]$$

(3.1)

$$\sum_{j=0}^{[k'/2]} \binom{k'}{2j+1} H_{3,3}^{3,2}\left[z \left| \begin{matrix} (1-\rho, \mu), \quad (1-\nu, 1), \quad (\beta+k'-2j-\rho, \mu) \\ (0, 1), (\beta+k'-2j-\rho-1, \mu), (\alpha+\beta+k'-\rho, \mu) \end{matrix} \right. \right]$$

$$\geq \sum_{j=0}^{[k'/2]} \binom{k'}{2j} H_{3,3}^{3,2}\left[z \left| \begin{matrix} (1-\rho, \mu), (1-\nu, 1), (1+\beta+k'-2j-\rho, \mu) \\ (0, 1), (\beta+k'-2j-\rho, \mu), (\alpha+\beta+k'-\rho, \mu) \end{matrix} \right. \right];$$

... (3.2)

$$\sum_{j=1}^{k} \frac{(-2k)_{2j}}{(2j)!} H_{3,3}^{3,2}\left[z \left| \begin{matrix} (1-\rho,\mu),(1-\nu,1),(1+\beta+2j-\rho,\mu) \\ (0,1),(\beta+2j-\rho,\mu),(\alpha+\beta+m-\rho,\mu) \end{matrix} \right. \right]$$

$$\geq \sum_{j=0}^{k-1} \frac{(-2k)_{2j+1}}{(2j+1)!} H_{3,3}^{3,2}\left[z \left| \begin{matrix} (1-\rho,\mu), \quad (1-\nu,1), \quad (2+\beta+2j-\rho, \quad \mu) \\ (0,1),(1+\beta+2j-\rho,\mu),(\alpha+\beta+m-\rho,\mu) \end{matrix} \right. \right]; \qquad \text{... (3.3)}$$

$$\sum_{j=1}^{[k'/2]} \frac{(-k')_{2j}}{(2j)!} H_{3,3}^{3,2}\left[z \left| \begin{matrix} (1-\rho,\mu),(1-\nu,1),(1+\beta+2j-\rho,\mu) \\ (0,1),(\beta+2j-\rho,\mu),(\alpha+\beta+m-\rho,\mu) \end{matrix} \right. \right]$$

$$\geq \sum_{j=0}^{[k'/2]} \frac{(-k')_{2j+1}}{(2j+1)!} H_{3,3}^{3,2}\left[z \left| \begin{matrix} (1-\rho,\mu), \quad (1-\nu,1), \quad (2+\beta+2j-\rho,\mu) \\ (0,1),(1+\beta+2j-\rho,\mu),(\alpha+\beta+m-\rho,\mu) \end{matrix} \right. \right] \qquad \text{... (3.4)}$$

$$\sum_{j=0}^{k} \sum_{r=0}^{n} \binom{2k}{2j} \frac{(-n)_r (\gamma-\alpha)_r}{r!(\gamma-2k-n)_r (\gamma-n-2j+r)_\lambda} \cdot$$

$$H_{2,2}^{1,2}\left[z \left| \begin{matrix} (1-\gamma-\lambda+n+2j-r,\mu),(1-\gamma,1) \\ (0,1), \quad (1-\nu+n+2j-\sigma-r,\mu) \end{matrix} \right. \right]$$

$$\geq \sum_{j=0}^{k-1} \sum_{r=0}^{n} \binom{2k}{2j+1} \frac{(-n)_r (\gamma-\alpha)_r}{r!(\gamma-2k-n)_r (\gamma-n-2j-1+r)_\lambda} \cdot$$

$$H_{2,2}^{1,2}\left[z \left| \begin{matrix} (2-\gamma-\lambda+n+2j-r,\mu),(1-\nu,1) \\ (0,1),(2-\gamma-\lambda+n+2j-\sigma-r,\mu) \end{matrix} \right. \right],$$

$$\gamma > \alpha > n+2k; \qquad \text{... (3.5)}$$

$$\sum_{j=0}^{[k'/2]} \sum_{r=0}^{n} \binom{k'}{2j+1} \frac{(-n)_r (\gamma-\alpha)_r}{r!(r-k'-n)_r \ (\gamma-n-2j+r-1)_\lambda} \cdot$$

$$H_{2,2}^{1,2}\left[z \left| \begin{matrix} (2-\gamma-\lambda+n+2j-r,\mu),(1-\nu,1) \\ (0,1),(2-\gamma-\lambda+n+2j-\sigma-r,\mu) \end{matrix} \right. \right],$$

$$\geq \sum_{j=0}^{[k'/2]} \sum_{\nu=0}^{n} \binom{k'}{2j} \frac{(-n)_n (r-\alpha)_r}{r!(r-k-n)_r (r(r-n-2j+r)_\lambda}$$

$$H_{2,2}^{1,2}\left[z \left| \begin{matrix} (1-r-\lambda+n+2j-r,\mu),(1-nu,1) \\ (0,1),(1-r-\lambda+n+2j-\sigma-r,\mu \end{matrix} \right. \right],$$

$\gamma > \alpha > n + k'$. ... (3.6)

Outline of proofs. To establish the inquality in (3.1), we first multiply both sides of (2.1) by $x^{\rho-1}(1-x)^{\beta+2k-\rho-1}\left[1+z\left(\frac{x}{1-x}\right)^{\mu}\right]^{-\nu}$, integrat with respect to x betwen limits 0 to 1 and then use (2.7) to arrive at the desired result in (3.1). The inequality in (3.2) is obtained by multiplying both sides of (2.2) by $x^{\rho-1}(1-x)^{\beta+k'-\rho-1}\left[1+z\left(\frac{x}{1-x}\right)^{\mu}\right]^{-\nu}$ instead and proceeding as above and using (2.7) therein.

The inequalities in (3.3), (3.4), (3,5) and (3.6) are established in a similar manner. We use the multiplying factors

$x^{\rho-1}(1-x)^{\beta-\rho-1}\left[1+z\left(\frac{x}{1-x}\right)^{\mu}\right]^{-\nu}$ in (2.3) and (2.4) for (3.3) and (3.4) and $x^{\gamma+\lambda-n}(1-x)^{\sigma-1}(1+z\,x^{\mu})^{-\nu}$ in (2.5), (2.6) for (3.5) , (3.6)) and make use of (2.7) and (2.8) respectively therein.

4. PARTICULAR CASES

1. In view of the formula

$$\underset{\nu\to\infty}{Lt}\left(1+\frac{x}{\nu}\right)^{-\nu} = e^{-x},$$

the inequalities in (3.1) and (3.3) reduce respectively to the following :

For $\alpha, \beta, \mu, \sigma, \rho, z > 0$; k., m and λ being positive integers, we have

$$\sum_{j=0}^{k}\binom{2k}{2j} H_{2,3}^{3,1}\left[z \,\middle|\, \begin{matrix}(1-\rho,\mu),(1+\beta+2k-2j-\rho,\mu)\\(0,1),(\beta+2k-2j-\rho,\mu),(\alpha+\beta+2k-\rho,\mu)\end{matrix}\right]$$

$$\geq \sum_{j=0}^{k-1}\binom{2k}{2j+1} H_{2,3}^{3,1}\left[z \,\middle|\, \begin{matrix}(1-\rho,\mu),(\beta+2k-2j-\rho,\mu)\\(0,1),(\beta+2k-2j-\rho-1,\mu),(\alpha+\beta+2k-\rho,\mu)\end{matrix}\right];$$

(4.1)

$$\sum_{j=1}^{k}\frac{(-2k)_{2j}}{(2j)\,!} H_{2,3}^{3,1}\left[z \,\middle|\, \begin{matrix}(1-\rho,\mu),(1+\beta+2j-\rho,\mu)\\(0,1),(\beta+2j-\rho,\mu),(\alpha+\beta+m-\rho,\mu)\end{matrix}\right]$$

$$\geq \sum_{j=0}^{k-1}\frac{(-2k)_{2j+1}}{(2j+1)!} H_{2,3}^{3,2}\left[z \,\middle|\, \begin{matrix}1-\rho,\mu),(i+\beta+2j-\rho,\mu)\\(0,1),(1+\beta+2j-\rho,\mu),(\alpha+\beta+m-p,\mu)\end{matrix}\right].$$

... (4.2)

2. If, in (3.5) and (3.6), we take $\mu = 1 = k'$, these yield respectively the following inequalities involving ${}_2F_1$ in view of the known relation Srivastava et al. [4,*p*.18, *eqn*. (2.6.3)] :

For k, n, σ and λ being positive integers and $\nu, z > 0$, we have :

$$\sum_{j=0}^{k} \sum_{r=0}^{n} \binom{2k}{2j} \frac{(-n)_r (\gamma-\alpha)_r}{r!\,(\gamma-2k-n)_r (\gamma-n-2j+r)_{\lambda+\sigma}} \; {}_2F_1\left(\begin{matrix} r+\lambda-n-2j+2, & \nu\ ; \\ \gamma+\lambda+\sigma-n-2j+r\ ; & \end{matrix} -z\right)$$

$$\geq \sum_{j=0}^{k-1} \sum_{r=0}^{n} \binom{2k}{2j+1} \frac{(-n)_r (\gamma-\alpha)_r}{r!\,(\gamma-2k-n)_r (\gamma-n-2j+r-1)_{\lambda+\sigma}} \cdot {}_2F_1\left(\begin{matrix} \gamma+\lambda-n-2j+r-1, & \nu\ ; \\ \gamma+\lambda+\sigma-n-2j+r-1\ ; & \end{matrix} -z\right).$$

$\gamma > \alpha > n + 2k$; ... (4.3)

$$\sum_{r=0}^{n} \frac{(-n)_r (\gamma-\alpha)_r}{r!\,(\gamma-n-1)_{r+\lambda+\sigma}} \; {}_2F_1\left(\begin{matrix} r+\lambda-n+r-1, & \nu\ ; \\ \gamma+\lambda+\sigma-n+r-1\ ; & \end{matrix} -z\right)$$

$$\geq \sum_{r=0}^{n} \frac{(-n)_r (\gamma-\alpha)_r}{r!\,(\gamma-n-1)_r (\gamma-n+r)_{\lambda+\sigma}} \; {}_2F_1\left(\begin{matrix} \gamma+\lambda-n+r, & \nu\ ; \\ \gamma+\lambda+\sigma-n+r\ ; & \end{matrix} -z\right),$$

$\gamma > \alpha > n + 1$. ... (4.4)

The inequalities in (3.1) to (3.6) and (4.3), (4.4) are also capable of yielding inequalities involving similar H-functions, ${}_1F_1$, ${}_2F_2$ and ${}_3F_2$, but these are not recorded here for lack of space.

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