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INTEGRATION OF CERTAIN PRODUCTS ASSOCIATED WITH THE MULTIVARIABLE *H*-FUNCTION AND A GENERAL CLASS OF POLYNOMIALS

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ABSTRACT

In the present paper we evaluate three finite integrals innvolving the products of Fox's *H*-function, the multivariable *H*-function and a general class of polynomials. On account of the most general nature of the functions invloved herein a very large number of known and new integrals involving simpler special functions and orthogonal polynomials follow as particular cases of our main integrals.

1. INTRODUCTION

Srivastava [5] introduced the general class of polynomials (see also Srivastava and Singh [9]) :

$$S_\lambda^w[x] = \sum_{\alpha=0}^{[\lambda/w]} \frac{(-\lambda)_{w\alpha}}{\alpha\,!} L_{\lambda,\alpha} x^\alpha, \ \lambda = 0, 1, 2, \ldots, \qquad \ldots(1.1)$$

where w is an arbitrary positive integer and the coefficients $L_{\lambda,\alpha}$ $(\lambda, \alpha \geq 0)$ are arbitrary constants real or complex. By suitably specializing the coefficeints $L_{\lambda,\alpha}$, the general class of polynomials can be reduced to a large spectrum of polynomials as cited in the papers referred to above.

The series representaion of Fox's H-function (see [1] and [4])

$$H_{p,q}^{m,n}\left[\begin{vmatrix}(e_p, E_p)\\(f_g, F_g)\end{vmatrix}\right] = \sum_{g=1}^{m} \sum_{s=0}^{\infty} (-1)^s \phi(\eta_s)\{s!\,F_g\}^{-1}, \qquad \ldots(1.2)$$

where

$$\phi(\eta_s) = \prod_{j=1, j\neq g} \Gamma(f_j - F_j\eta_s) \prod_{j=1}^{n} \Gamma(1 - e_j + E_j\eta_s)$$
$$\cdot\left\{\prod_{j=m+1}^{q} \Gamma(1 - f_j - F_j\eta_s) \prod_{j=n+1}^{p} \Gamma(e_j - E_j\eta_s)\right\}^{-1}$$

and $\eta_s = (f_g + s)/F_g$ $\qquad \ldots(1.3)$

For the multivariable H-function defined by Srivastava and Panda ([6] and [7]; see also [8]), we derive three finite integrals in the next section.

2. THE MAIN INTEGRALS

The following integrals have been established in this section :

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1}\{hx+k(1-x)\}^{-2\rho}$$

$$\cdot\ {}_2F_1\left[u, v; (u+v+1)/2; hx\{hx+k(1-x)\}^{-1}\right]$$

$$\cdot\ H^{m,n}_{p,q}\left[z\{hkx(1-x)\}^{\sigma}\{hx+k(1-x)\}^{-2\sigma}\ \Big|\ \begin{matrix}(e_p, E_p)\\(f_q, F_q)\end{matrix}\right]$$

$$\cdot\ S^{w}_{\lambda}\left[z\{hkx(1-x)\}^{\mu}\{hx+k(1-x)\}^{-2\mu}\right]$$

$$H^{0,B:(M',N');\dots;(M^{(r)},N^{(r)})}_{A,C:(P',Q');\dots;(P^{(r)},Q^{(r)})}\left(\frac{z_1\{hkx(1-x)\}^{\sigma_1}}{\{hx+k(1-x)\}^{2\sigma_1}},\dots,\frac{z_r\{hkx(1-x)\}^{\sigma_r}}{\{hx+k(1-x)\}^{2\sigma_r}}\right)dx w$$

here

$$= \sum_{\alpha=0}^{[\lambda/w]}\sum_{g=1}^{m}(-1)^s\phi(\eta_s)z^{\eta_s}\{s!F_g\}^{-1}\frac{(-\lambda)_{w\alpha}}{\alpha!}L_{\lambda,\alpha}$$

$$\cdot\ 2^{1-\dots\sigma-\eta_s-2\dots}\pi\,\Gamma\{(u+v+1)/2\}(hk)^{\rho}\,\Gamma\{(u+1)/2\}\,\Gamma\{(v+1)/2\}^{-1}$$

$$\cdot\ H^{0,B+2:(M',N');\dots;(M^{(r)},N^{(r)})}_{(A+2,C+2:(P',Q');\dots;(P^{(r)},Q^{(r)})}\left(\begin{matrix}[1-\rho-\sigma\eta_s-\mu\alpha:\sigma_1,\dots,\sigma_r],\\ \dots],\\ [-\sigma-\sigma\eta_s-\mu\alpha+(u+v+1)/2:\sigma_1,\dots,\sigma_r],[(a):\theta',\dots,\theta^{(r)}]:\\ [-\rho-\sigma\eta_s-\mu\alpha+(u+1)/2:\sigma_1,\dots,\sigma_r],[-\rho-\sigma\eta_s-\mu\alpha+(v+1)/2:\sigma_1,\dots,\sigma_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1 4^{-\sigma_1},\dots,z_r 4^{-\sigma_r}\right) \qquad \dots(2.1)$$

where $\sigma > 0, \mu > 0, \sigma_i > 0, Re\ \{(1-u-v)/2\} > 0$,

$Re\ (\rho + \sigma f_l / F_l + \sum_{i=1}^{r} \sigma_i\ a_j^{(i)} / \delta_j^{(i)}) > 0,\ l = 1, \dots, m\ ;\ j = 1, \dots, M^{(i)}$,

$T > 0$, $|\arg z| < T\pi/2$, $|\arg z_i| < T_i\pi/2$, h, k are non-zero constant and $\{hx + k(1-x)\}$ is non-zero where $0 \le x \le 1$, w is an artitrary positive integer and the coefficents $L_{\lambda,\alpha}$ $(\lambda, \alpha \ge 0)$ an arbitrary constants real or complex

$$\left(T = \sum_{1}^{n} E_i - \sum_{n+1}^{p} E_i + \sum_{1}^{m} F_i - \sum_{m+1}^{q} F_i\right).$$

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-v}\{hx+k(1-x)\}^{-2\rho+-1}$$

$$_2F_1\left[u-1-u : v : hx\{hx+k(1-x)\}^{-1}\right]$$

$$.\ H_{p,q}^{m,n}\left[z\{hkx(1-x)\}^{\sigma}\{hx+k(1-x)\}^{-2\sigma}\left|\begin{matrix}(e_p, E_p)\\(f_q, F_q)\end{matrix}\right.\right]$$

$$.\ S_\lambda^w\left[\{hkx(1-x)\}^{\mu}\{hx+k(1-x)\}^{-2\mu}\right]$$

$$.\ H_{A,C:(P',Q');\ldots;(P^{(r)},Q^{(r)})}^{0,B:(M',N');\ldots;(M^{(r)},N^{(r)})}\left(\frac{z_1\{hkx+(1-x)\}^{\sigma_1}}{\{hx+k(1-x)\}^{2\sigma_1}},\ldots,\frac{z_r\{hkx+(1-x)\}^{\sigma_r}}{\{hx+k(1-x)\}^{2\sigma_r}}\right)dx$$

$$=\sum_{\alpha=0}^{[\lambda/w]}\sum_{g=1}^{m}\sum_{s=0}^{\infty}(-1)^s\phi(\eta_s)z^{\eta_s}\{s!F_g\}^{-1}\frac{(-\lambda)_{w\alpha}}{\alpha!}L_{\lambda,\alpha}$$

$$.\ (hk)^{\sigma\eta_s+\mu\alpha}2^{1-2\sigma-2\rho\eta_s-2\mu\alpha}\Gamma(v)\Pi\{\Gamma(u+v)/2\}\Gamma\{(1-u+v)/2\}^{-1}$$

$$H_{A+2,C+2:(P',Q');\ \ ;Q^{(r)},P^{(r)}}^{0,\ \ B+2:(M',N');\ldots;(M^{(r)},N^{(r)})}\left(\begin{matrix}[1-\rho-\sigma\eta_s-\mu\alpha:\sigma_1,\ldots,\sigma_r],\\ [(c):\psi',\ldots,\psi^{(r)}],\end{matrix}\right.$$

$$[v-\rho-\sigma\eta_s-\mu\alpha:\sigma_1,\ldots,\sigma_r],[(a):\theta',\ldots,\theta^{(r)}]:$$

$$[-\rho-\sigma\eta_s-\mu\alpha+(1-u+v)/2:\sigma_1,\ldots,\sigma_r],[1-\rho-\sigma\eta_s-\mu\alpha+(u+v)/2:\sigma_1,\ldots,\sigma_r],[(b')$$

$$[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\quad z_1(hk)^{\sigma_1}4^{\sigma_1},\ldots,z_r(hk)^{\sigma_r}4^{\sigma_r}\qquad \ldots(2,2)$$

where $\sigma > o$, $\mu > 0$, $\sigma_i > 0$, $\text{Re}(v) > 0$,

$\text{Re}(\rho+\sigma f_l/F_l+\sum_{i=1}^{r}\sigma_r\, a_j^{(i)}/delta_j^{(i)}) > 0$, $l=1,\ldots,m$; $j=1,\ldots,M^{(i)}$,

$|\arg z| < T\pi/2$, $|\arg(z_i)| < T_i\pi/2$, $T>0$, $T_i>0$ and h,k are non-zero constants and $\{hx+k(1-x)\}$ is non-zero where $0\le x\le 1$, w is an arbitrary positive integer and the coefficients $L_{\lambda,\alpha}(\lambda,\alpha\ge 0)$ are arbitrary constants real or complex.

$$.\int_0^1 t^{u-1}(1-t)^{a-2u}(1+vt)^{u-a-1}\,{}_2F_1\left[a,b;1+a-b;(1+v)t\{(1+vt)\}\right]$$

$$.\ H_{p,q}^{m,n}\left[z\left\{\frac{4(1+v)(t+vt^2)}{(1-t)^2}\right\}^h\left|\begin{matrix}(e_p, E_p)\\(f_q, F_q)\end{matrix}\right.\right].\ S_\lambda^w\left[\left\{\frac{4(1+v)(t+vt^2)}{(1-t^2)}\right\}^{\mu}\right]$$

$$\cdot\ H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{A,C:(p',Q');\ldots;(P^{(r)},Q^{(r)})}\left(z_1,\ldots,z_i\left\{\frac{4(1+v)(t+vt^2)}{(1-t)^2}\right\}^{k_i},\ldots,z_r\right)dt$$

$$=\sum_{\alpha=0}^{[\lambda/w]}\sum_{g=1}^{m}\sum_{s=0}^{\infty}(-1)^s\phi(\eta_s)z^{\eta_s}\{s!\,F_g\}^{-1}\frac{(-\lambda)_{w\alpha}}{\alpha!}L_{\lambda,\alpha}$$

$$.\ 2^{a-2u}(1+v)^{-u}\,\Gamma(1+a/2)\,\Gamma(1+a-b)\{\sqrt{\pi}\ \Gamma(1+a)\,\Gamma(1-b+a/2)\}^{-1}$$

$$\cdot\ H^{0,0:(M',N');\ldots;(m^{(i)}+2,N^{(i)}+1;\ldots;(M^{(r)},N^{(r)})}_{A,C:(P',Q');\ldots;(P^{(i)}+2,Q^{(i)}+2);\ldots;(P^{(r)},Q^{(r)})}\left(\begin{matrix}[(a):e',\ldots,e^{(r)}]:\\ [(c):\psi',\ldots,\psi^{(r)}]:\end{matrix}\right.$$

$$[(b'):\phi'];\ldots;[1-u-h\eta_s-\mu\alpha:k],[(b^{(i)}):\phi^{(i)}],$$

$$[(d'):\delta'];\ldots;[-u-h\eta_s-\mu\alpha+(1+a)/2:k],[1-u-h\eta_s-\mu\alpha+\frac{a}{2}-b:k],$$

$$\left.\begin{matrix}[1+a-b-u-h\eta_s-\mu\alpha:k];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(d^{(i)}):\delta^{(i)}]\qquad\qquad;\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\ldots,z_r\right),\qquad \ldots(23)$$

where $v>-1, h>0, k>0, \mu>0$, $\mathrm{Re}(a-2u-2b+2)>0$,

$Re(u+h\,f_l/F_l+k\,d_j^{(i)}/\delta_j^{(i)})>0,\quad l=1,\ldots,m;\ j=1,\ldots,M^{(i)}$,

$Re\left(a-2u-2h(e_{l'}-1)/E_{l'}-2k\,(1-b_j^{(i)}/\phi_j^{(i)})+1\right)>0,\ l'=1,\ldots,n$;

$j=1,\ldots,N^{(i)}$, $|\,arg\,z\,|<T\pi/2$, $|\,arg\,(z_i)\,|<T_i\,\pi/2$, $T>0$,

$T_i>0, i=1,\ldots,r$ and w is an arbitrary positive integer and the coefficients $L_{\lambda,\alpha}$ $(\lambda,\alpha\geq 0)$ are arbitrary constants real of complex.

Proof : To establish (2.1), express the general class of polynomials occurring in the integrand of (2.1) in series form given by (1.1) and then interchange the order of summation and integration (which is permissible under the conditions stated above), evaluate the inner integral with the help of a result of Chaurasia ([2],p.67,eq.(2.1)), we arrive at the desired result.

The proofs of the integrals (2.2) and (2.3) can be developed by proceeding on similar lines with the help of known integrals ([2],p.67,eq.(2.2) and p.68,eq.(2.3)).

3. SPECIAL CASES

(i) Giving suitable values to the parameters and using a result ([6],p.138), the results (2.1) to (2.3) can be reduced to the following integral formulae involving the Lauricella's fucntions :

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1}\{hx+k(1-x)\}^{-2\rho}\ {}_2F_1\left[u,v;(u+v+1)/2;\frac{hx}{hx+k(1-x)}\right]$$

$$. H_{p,q}^{m,n}\left[\frac{z\{hkx(1-x)\}^{\sigma}}{\{hx+k(1-x)\}^{2\sigma}}\ \middle|\begin{matrix}(e_p, E_p)\\(f_q, F_q)\end{matrix}\right] . S_{\lambda}^{w}\left[\frac{\{hkx(1-x)\}^{\mu}}{\{hx+k(1+x)\}^{2\mu}}\right]$$

$$. F_{C:D';\ldots;D^{(r)}}^{A:B';\ldots;B^{(r)}}\left(\frac{z_r\{hkx(1-x)\}^{\sigma_1}}{\{hx+k(1-x)^{2\sigma_1}}, \ldots, \frac{z_r\{hkx(1-x)\}^{\sigma_r}}{\{hx+k(1-x)^{2\sigma_r}}\right) dx$$

$$= \sum_{\alpha=0}^{[\lambda/w]} \sum_{g=1}^{m} \sum_{s=0}^{\infty} (-1)^s \phi(\eta_s) z^{\eta_s} \{s!F_g\}^{-1} \frac{(-\lambda)_{w\alpha}}{\alpha!} L_{\lambda,\alpha} \Pi\, \Gamma(\rho+\sigma\eta_s+\mu\alpha)$$

$$. \Gamma\{\rho+\sigma\eta_s+\mu\alpha+(1-u-v)/2\} \Gamma\{(u+v+1)/2\} . 2^{1-2\rho-2\sigma\eta_s-2\mu\alpha}$$

$$. \Big[(hk)^{\rho} \Gamma\{(1+u)/2\} \Gamma\{(1+v)/2\} \Gamma\{\rho+\eta_s$$

$$+\mu\alpha+(1-u)/2\} \Gamma\{\rho+\sigma\eta_s+\mu\alpha+(1-v)/2]^{-1}$$

$$. F_{C+2:D';\ldots;D^{(r)}}^{A+2:B';\ldots;B^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}], [\rho+\sigma\eta_s+\mu\alpha:\sigma_1,\ldots,\sigma_r],\\ [(c):\psi',\ldots,\psi^{(r)}], [\rho+\sigma\eta_s+\mu\alpha+(1-u)/2:\sigma_1,\ldots,\sigma_r],\end{matrix} [\rho+\sigma\right.$$

$$[\rho+\sigma\eta_s+\mu\alpha+(1-v)/2:\sigma_1,\ldots,\sigma_r] : [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];$$

$$\left. ,\qquad z_1(4)^{-\sigma_1},\ldots,z_r(4)^{-\sigma_r}\right), \qquad \ldots (3.1)$$

provided that $\sigma > 0, Re(\rho+(1-u-v)/2+\sigma f_l/F_l) > 0, l = 1, \ldots, m,$ $|\arg z| < T\pi/2, T > 0, h, k$ are non-zero constants and $\{hx+k(1-x)\}$ is non-zero when $0 \le x \le 1$, w is an arbitrary postive constant and the coefficients $L_{\lambda,\alpha}$ $(\lambda, \alpha \ge 0)$ are arbitrary constants real or complex and the resulting multiple series converges.

$$\int_0^1 x^{l-1}(1-x)^{\rho-v}\{hx+k(1-x)\}^{-2\rho+v-1}$$

$$_2F_1\left[u, 1-u; v; hx\{hx+k(1-x)\}^{-1}\right]$$

$$H_{p,q}^{m,n}\left[\frac{z\{hkx(1-x)\}^{\sigma}}{\{hx+k(1-x)\}^{2\sigma}}\ \middle|\begin{matrix}(e_p, E_p)\\(f_q, F_q)\end{matrix}\right] . S_{\lambda}^{w}\left[\frac{\{hkx(1-x)\}^{\mu}}{\{hx-k(1-x)\}^{2\mu}}\right]$$

$$F_{C:D';\ldots;D^{(r)}}^{A:B';\ldots;B^{(r)}}\left(\frac{z_1\{hkx(1-x)\}^{\sigma_1}}{\{hx+k(1-x)\}^{2\sigma_1}}, \ldots, \frac{z_r\{hkx(1-x)\}^{\sigma_r}}{\{hx+k(1-x)\}^{2\sigma_r}}\right) dx$$

$$= \sum_{\alpha=0}^{[\lambda/w]} \sum_{g=1}^{m} \sum_{s=0}^{\infty} (-1)^s \phi(\eta_s) z^{\eta_s} \{s!F_g\}^{-1} \frac{(-\lambda)_{w\alpha}}{\alpha!} L_{\lambda,\alpha} (hk)^{\sigma\eta_s+\mu\alpha}$$

$$\cdot\ \pi\ 2^{1-2\rho-2\sigma\eta_s-2\mu\alpha}\,\Gamma(v)\,\Gamma(\rho+\sigma\eta_s+\mu\alpha)\,\Gamma(\rho+\sigma\eta_s+\mu\alpha-v+1)$$

$$\cdot\ \Big[\,\Gamma\{(u+v)/2\}\,\Gamma\{(1-u+v)/2\}\,\Gamma\{\rho+\sigma\eta_s+\mu\alpha+(u-v+1)/2\}$$

$$\cdot\ \Gamma\{\rho+\sigma\eta_s+\mu\alpha-(u+v)/2\}\Big]^{-1}$$

$$\cdot\ F^{A+2:B',\dots,B^{(r)}}_{C+2:D',\dots,D^{(r)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(r)}],\ [\rho+\sigma\eta_s+\mu\alpha:\sigma_1,\dots,\sigma_r],\\ [(c):\psi',\dots,\psi^{(r)}],\ [\rho+\sigma\eta_s+\mu\alpha+(u-v+1)/2;\sigma_1,\dots,\sigma_r],\end{matrix}\right.$$

$$[\rho+\sigma\eta_s+\mu\alpha-v+1:\sigma_1,\dots,\sigma_r]:[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}];$$

$$[\rho+\sigma\eta_s+\mu\alpha-(u+v)/2:\sigma_1,\dots,\sigma_r]:[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];$$

$$\left. z_1\frac{(hk)^{\sigma_1}}{4^{\sigma_1}},\dots,z_r\frac{(hk)^{\sigma_r}}{4^{\sigma_r}}\right), \qquad \dots(3.2)$$

provided that $\sigma > 0$, $\mathrm{Re}(v) > 0$; $\mathrm{Re}(\rho+\sigma f_l/F_l) > l = 1, \dots, m$, $|\arg z| < T\pi/2$, $T > 0$, h, k are non-zero constants and $\{hx + k(1-x)\}$ is non-zero where $0 \le x \le 1$, w is an arbitrary postive constants and the coefficients $L_{\lambda,\alpha}$ $(\lambda, \alpha \ge 0)$ are arbitrary constants real or complex and the resulting multiple series converges. Similarly the result in (2.3) can be reduced to an integral involving the Lauricella's functions as a special case.

(ii) When $\lambda = 0$, the integrals in (2.1) to (2.3), (3.1) and (3.2) reduce to the integrals recently obtained by Chaurasia [2].

The importnace of our integrals possess mainifold generality. At the outset, we recall that in view of the generality of the polynomials $S^{w}_{\lambda}[x]$, on suitable specializing the coefficients $L_{\lambda,\alpha}$, and making a free use of the special cases of $S^{w}_{\lambda}[x]$ listed by Srivastava and Singh [9], our integrals can be reduced to a large number of integrals involving generalized Hermite polynomials, Hermite polynomials, Jacobi polynomials and its various special cases, Laguerre polynomials, Bessel polynomials, Gould-Hopper polynomials, Brafman polynomials and their various combinations.

Secondly, by specializing the various parameters and variables in Fox's H-function and in the multivariable H-function, we can obtain from our results, several integrals involving a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of E, F, G and H functions of one and several variables. Thus our results would at once yield a very large number of integrals involving a large variety of polynomials and various special functions occurring in the problems of mathmetical analysis, applied mathematics and mathematical physics.

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