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(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

THE DISTRIBUTON OF THE QUOTIENT OF PRODUCTS OF RATIONAL POWERS OF INDEPENDENT RANDOM VARIABLES

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ABSTRACT

In this paper an attempt has been made to present the distribution of the quotient of products of rational powers of independent Stochastic varibales with probability denstiy function associated with Beta distribution involving Fox's H- fucntion.

1. Introduction

A number of statistical distribuions have been studied from time to time by several authors. For example, Methai and Saxena [1], and Srivastava and Singhal [3], studied a general class of distributions, whose probability density fucntion involves a hypergeometric funtion ${}_2F_1$ and Fox's H-function respectively. Further, Exton [2] and Agrawal [5] considered the family of distributions which have the probability density function in terms of the product of several generalized hypergeometric funcions ${}_pF_q$ and the H-function of several varibales respectively. In this paper we have considered the probability denstiy fucntion associated with Beta distribution involving Fox's H-function, to study the distribution of the quotient of products of rational powers of independent random variables.

Let x_i $(i=1, ..., r)$ be r independent random variables such that their probability density fucntion, which follows from [5,p.93,eq.(2.1)] is given by

$$f_i(x_i) = \begin{cases} L_i x_i^{\lambda_i - 1}(1-x_i)^{\gamma_i} H_{p_i, q_i}^{m_i, n_i}\left[a x_i^{u_i}(1-x_i)^{v_i} \middle| \begin{matrix} (a_j^{(i)}, \alpha_j^{(i)})_{1,p_i} \\ (b_j^{(i)}, \beta_j^{(i)} 1)_{1,q_i} \end{matrix} \right], \\ 0, \quad \text{elsewhere} \end{cases}$$

$$0 \le x_i \le 1, i = 1, ..., r \qquad \text{... (1.1)}$$

where

$$L_i^{-1} = H_{p_i+2, q_i+1}^{m_i, n_i+2}\left[\begin{matrix} (1-\lambda_i, u_i), (-\gamma_i, v_i), (a_j^{(i)}, \alpha_j^{(i)})_{1,p_i} \\ (b_j^{(i)}, \beta_j^{(i)})_{1,q_i}, (-\lambda_i-\gamma_i, (-\lambda_i-\gamma_i, u_i+v_i) \end{matrix} \right] \qquad \text{... (1.2)}$$

$H_{p,q}^{m,n}[x]$ denotes the well-known Fox's H-function [4] and the following conditions are satisfied

(i) $u_i, v_i \geq 0$ (not both zero simultaneously)

(ii) $\lambda_i + u_i \min\limits_{1 \leq j \leq m_i} \left(b_j^{(i)}/\beta_j^{(i)}\right) > 0$

(iii) $\gamma_i + v_i \min\limits_{1 \leq j \leq m_i} \left(b_j^{(i)}/B_j^{(i)}\right) + 1 > 0$

(iv) $A = \sum\limits_{j=1}^{m_i} \beta_j^{(i)} - \sum\limits_{j=m_i+1}^{q_i} \beta_j^{(i)} + \sum\limits_{j=1}^{n_i} \alpha_j^{(i)} - \sum\limits_{j=n_i+1}^{p_i} \alpha_j^{(i)} > 0$

(v) The parameters involved in (1.1) are so restricted that $f_i(x_i)$ remains positive for $0 \leq x_i \leq 1$, $(i = 1, ..., r)$ ... (1.3)

2. Distribution of the quotient of products of rational powers of several random variables

The theorem given below gives us the distribution of the quotient of products of rational powers of several independent random variables associated with the probability denstiy function defined by (1.1).

Theorem : Let $x_i\ (i = 1, ..., r)$ be r independent random variables, where x_i has the probabilty density function defined by (1.1). Then the probability denstiy function $h(u)$ of

$$U = \frac{x_1^{p_1} \cdots x_n^{p_n}}{x_{n+1}^{p_{n+1}} \cdots x_r^{p_r}} \ (1 \leq n \leq r) \text{ with } p_i \geq o\ (i = 1, ..., r) \quad \text{... (2.1)}$$

is given by

$$h(u) = \prod_{i=1}^{r} \left\{ L_i \sum_{j=1}^{m_i} \sum_{\mu_i=0}^{\infty} \psi_i(m_i, \mu_i, \rho_{ij}) \frac{(-1)^{\mu_i} a_i^{ij}}{\mu_i!\, \beta_j^{(i)}} \Gamma(1 + \gamma_i + v_i \rho_{ij}) \right\}.$$

$$. H_{r,r}^{n,r-n}\left[U \left| \begin{matrix} (1 - \lambda_i - u_i \rho_{ij}, p_i)_{n+1,r}, (1 + \gamma_i + \lambda_i + u_i \rho_{ij} + v_i \rho_{ij}, p_i)_{1,n} \\ (\lambda_i + u_i \rho_{ij}, p_i)_{1,n}, (-\gamma_i - \lambda_i - \mu_i \rho_{ij} - v_i \rho_{ij}, p_i)_{n+1,r} \end{matrix} \right. \right] \quad (2.2)$$

where

$$\rho_{ij} = \frac{b_j^{(i)} + \mu_i}{\beta_j^{(i)}} \text{ and } \psi_i(m_i, \mu_i, \rho_{ij}) =$$

$$\frac{\prod_{\substack{h=1\\h\neq j}}^{m_i} \Gamma\left(b_n^{(i)} - \beta_h^{(i)} \rho_{ij}\right) \prod_{h=1}^{n_i} \Gamma\left(1 - a_h^{(i)} + \alpha_h^{(i)} \rho_{ij}\right)}{\prod_{h=m_i+1}^{q_i} \Gamma\left(1 - b_h^{(i)} + \beta_h^{(i)} \rho_{ij}\right) \prod_{h=n_i+1}^{p_i} \Gamma\left(a_h^{(i)} - \alpha_h^{(i)} \rho_{ij}\right)} \quad \ldots (2.3)$$

The theorem is valid provided that $\left\{\lambda_i \pm sP_i + u_i \min_{1\le j\le m_i} \left(b_j^{(i)}/\beta_j^{(i)}\right)\right\} > 0$ and the conditions (i), (iii), (iv) and (v) of (1.3) are satisfied.

proof : Let

$$V = \log U = \sum_{i=1}^{n} \log x_i^{P_i} - \sum_{i=n+1}^{r} \log x_i^{P_i} = \sum_{i=1}^{n} \log Y_i - \sum_{i=r+1}^{r} \log Y_i.$$

We first obtain the probability function of $Y_i = X_i^{P_i}$. For this we make use of a known theorem [7, p. 70, eg. (2)], with (1.1) and easily get the p.d.f. $G_i(Y_i)$ of Y_i as follows :

$$G_i(Y_i) = \begin{cases} L_i P_i^{-1} Y_i^{\lambda/P_i - 1} \left(1 - Y_i^{1/P_i}\right)^{\gamma_i} H_{p_i, q_i}^{m_i, n_i}\left[a_i Y_i^{u_i/P_i}\left(1 - Y_i^{1/P_i}\right)^{v_i} \,\middle|\, \begin{matrix} \left(a_j^{(i)}, \alpha_j^{(i)}\right)_{1, p_i} \\ \left(b_j^{(i)}, \beta_j^{(i)}\right)_{1, q_i} \end{matrix}\right], & 0 \le Y_i \le 1 \\ 0, \text{ elsewhere} & \end{cases}$$

$$i = 1, \ldots, r \quad \ldots (2.4)$$

Next, to obtain the probability density fucntion of U, we use the method of Laplace transform and its inverse.

Let $\phi_v(s)$ denote the Laplace transform of V, then

$$\phi_v^{(s)} = E\left(e^{-sv}\right) = E\left(\prod_{i=1}^{n} Y_1^{-s} \prod_{i=n+1}^{r} Y_i^{S}\right)$$

$$= \prod_{i=1}^{n}\left[\int_0^\infty G_i(y_i)\, Y_i^{-s}\, dy_i\right] \prod_{i=n+1}^{r}\left[\int_0^\infty G_i(Y_i)\, Y_i^{S}\, dY_i\right] \quad \ldots (2.5)$$

Substituting the value of $G_i(Y_i)$ from equaion (2.4) in (2.5), changing the variable of integration slightly and evaluating the integrals thus obtained with the help of [4,p.61,eq.(5.2.3)], we get

$$\phi_v(s) = \prod_{i=1}^{n}\left\{L_i H_{p_i+2, q_i+1}^{m_i, n_i+2}\left[a_i \,\middle|\, \begin{matrix} (1 - \lambda_i + sp_i, u_i), (-\gamma_i, v_i), \left(a_j^{(i)}, \alpha_j^{(i)}\right)_{1, p_i} \\ \left(b_j^{(i)}, \beta_j^{(i)}\right)_{1, q_i} (-\lambda_i + sp_i - \gamma_i, u_i + v_i) \end{matrix}\right]\right\},$$

$$\prod_{i=n+1}^{r}\left\{L_i H_{p_i+2,q_i+1}^{m_i,n_i+2}\left[a_i\left|\begin{matrix}(1-\lambda_i-sp_i,u_i),(-\gamma_i,v_i),(a_j^{(i)},\alpha_j^{(i)})_{1,p_i}\\(b_j^{(i)},\beta_j^{(i)})_{1,q_i}(-\lambda_i-sp_i-\gamma_i,u_i+v_i)\end{matrix}\right.\right]\right\}, \quad \text{... (2.6)}$$

Now we express each H-function invloved in (2.6) into series form [4, p. 12, eq. (2.2.4)] and collect the terms containing s, viz.

$$\prod_{i=1}^{n}\left\{\frac{\Gamma(\lambda_i-sp_i+u_i\rho_{ij})}{\Gamma(1+\gamma_i+\lambda_i+u_i\rho_{ij}+v_i\rho_{ij}-sp_i)}\right\}\prod_{i=n+1}^{r}\left\{\frac{\Gamma(\lambda_i+sp_i+u_i\rho_{ij})}{\Gamma(1+\gamma_i+\lambda_i+u_i\rho_{ij}+v_i\rho_{ij}+sp_i)}\right\} \quad \text{... (2.7)}$$

On taking the inverse Laplace transform of the expression (2.7) and substituting the value thus obtained in the expanded series of (2.6), we shall arrive at the desired result (2.2) after a little simplification.

3. Particular Cases

The probability density function given by (1.1) is quite general in nature due to the presence of Fox's H-function and a number of known distributions can be derived as specialized or limiting cases of this distribution. For example, if we take $\alpha_j^{(i)}=\beta_j^{(i)}=1, m_i=1, n_i=p_i$ and $q_i\to q_i+1,\ a_j^{(i)}\to 1-a_j^{(i)},\ b_j^{(i)},\ \to 1-b_j^{(i)}$ in (1.1), (1.2) and (2.2), we arrive at the following :

Corollary 1 : If

$$f_i(x_i)=\begin{cases}M_i x_i^{\lambda_i-1}(1-x_i)^{\gamma_i}\,{}_{p_i}F_{q_i}\left[a_j^{(i)};b_j^{(i)};-a_i x_i^{u_i}(1-x_i)^{v_i}\right], & 0\le x_i\le 1\\ 0, & \text{elsewhere}\end{cases}$$

$$i=1,\ldots,r \quad \text{... (3.1)}$$

then the density function of U as defined by (2.1) is given by

$$h_1^{(u)}=\prod_{i=1}^{r}\left\{M_i\sum_{\mu_i=0}^{\infty}\left[\prod_{j=1}^{p_i}\Gamma\left(a_j^{(i)}+\mu_i\right)\right]\left[\prod_{j=1}^{q_i}\Gamma(b_j^{(i)}+\mu_i)\right]^{-1}\right.,$$

$$\times\frac{(-a_i)^{\mu_i}}{\mu_i!}\Gamma(1+\gamma_i+v_i\mu_i)\ H_{r,r}^{n,r-n}[U]$$

$$\begin{matrix}(1-\lambda_i-u_i\mu_i,p_i)_{n+1,r}(1+\gamma_i+\lambda_i+u_i\mu_i+v_i\mu_i,p_i)_{1,n}\\(\lambda_i+\mu_i\ \mu_i,P_i)_{1,n}(-\gamma_i-\lambda_i-u_i\mu_i-v_i\mu_i,p_i)_{n+1,r}\end{matrix} \quad \text{... (3.2)}$$

where

$$M_i^{-1}=H_{p_i+2,q_i+2}^{1,p_i+2}\left[a_i\left|\begin{matrix}(1-\lambda_i,u_i),(-\gamma_i,v_i),(1-a_j^{(i)},1)_{1,p_i}\\(0,1),(1-b_j^{(i)},1)_{1,q_i}(-\lambda_i-\gamma_i,u_i+v_i)\end{matrix}\right.\right]. \quad \text{... (3.3)}$$

and the conditions easily obtainable from those stated with the main theorem are satisfied

On taking $\gamma_i = 0$ and $v_i = 0$ in (1.1) and (2.2), we arrive at the following :

Corollary 2 : If

$$f_i(x_i) = \begin{cases} K_i x_i^{\lambda_i - 1} H_{p_i, q_i}^{m_i, n_i}\left[a_i x_i^{u_i} \left| \begin{matrix} (a_j^{(i)}, \alpha_j^{(i)})_{1, p_i} \\ (b_j^{(i)}, \beta_j^{(i)})_{1, q_i} \end{matrix} \right. \right], 0 \le x_i \le 1 \\ 0, \text{ elsewhere} \end{cases}$$

$$i = 1, \ldots, r \qquad \ldots (3.4)$$

then the density function of U is given by

$$h_2(u) = \prod_{i=1}^{r} \left\{ K_i \sum_{j=1}^{m_i} \sum_{\mu_i = 0}^{\infty} \psi_i (m_i, \mu_i, \rho_{ij}) \frac{(-1)^{\mu_i} a_i^{\rho_{ij}}}{\mu_i !\, B_j^{(i)}} \right\} \cdot$$

$$\cdot H_{r, r}^{n, r-n}\left[u \left| \begin{matrix} (1 - \lambda_i - u_i \rho_{ij}, p_i)_{n+1, r} (1 + \lambda_i + u_i \rho_{ij}, p_i)_{1, n} \\ (\lambda_i + u_i \rho_{ij}, p_i)_{1, n} (-\lambda_i - u_i \rho_{ij}, p_i)_{n+1, r} \end{matrix} \right. \right] \qquad \ldots (3.5)$$

where

$$K_i^{-1} = H_{p_i+1, q_i+1}^{m_i, n_i+1}\left[a_i \left| \begin{matrix} (1 - \lambda_i, u_i), (a_j^{(i)}, \alpha_j^{(i)})_{1, p_i} \\ (b_j^{(i)}, \beta_j^{(i)})_{1, q_i}, (-\lambda_i, u_i) \end{matrix} \right. \right] \qquad \ldots (3.6)$$

and the conditions easily obtainable from those of the main theorem are satisfied.

(iii) On putting $p_i = 1$ $(i = 1, \ldots, r)$ and $X_2 \ldots X_n = 1 = X_{n+2} \ldots X_r$ in (2.2) we obtain a result essentially similar to the one variable analogue of the result given by Goyal and Audich [6,p.9,eq.(2.12)].

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