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(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

FIXED POINT THEOREM FOR THREE MAPPINGS ON METRIC SPACE

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ABSTRACT

Bhagwat and Singh [1] have extended the result of Das and Gupta [2] for pair of mappings. In this paper we prove a fixed point theorem for three mappings on a metric space, giving a new type of rational inequality like that of Bhagwat and Singh [1].

Definition : Let S, T and P be three self mappings of a metric space (X,d), then we say that $\{S, T\}$ is a weakly commuting pair of mappings with respect to mapping p,

if $d(PSPx, TPx) < d(SPPz, TPx)$ (i)

and $d(SPx, PTPx) < d(SPx, TPPx)$ (ii)

for all x in X.

Now we establish the following

Theorem : Let (x, d) be a complete metric space, let P,S and T be continuous self mappings on x such that $\{S, T\}$ is a weakly commuting pair of mappings with respect to mapping P and

$$d(SPx, TPy) \leq \frac{d(x, SPx)\, d(y, TPy) + d(y, SPx) d(x, TPy)}{d(x, SPx) + d(x, TPy)}$$

for alll x, y in X. If for some $x_0 \in X$, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to a point $z \in X$, then z is a common fixed point of SP and is a unique common fixed point of S,P and T.

Proof : Let $x_0 \in X$ such that

$$x_{2n+1} = SPx_{2n}, x_{2n} = TPx_{2n-1}, x_{2n+2} = TPx_{2n+1}.$$

we have

$$d(x_{2n+1}, x_{2n+2}) = d(SPx_{2n}, TPx_{2n+1})$$

$$\leq \frac{d(x_{2n}, SPx_{2n})\, d(x_{2n+1}, TPx_{2n+1}) + d(x_{2n+1}, SPx_{2n})\, d(x_{2n}, TPx_{2n+1})}{d(x_{2n}, SPx_{2n} + d(x_{2n}, TPx_{2n+1})}$$

$$= \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}), x_{2n+1}\, d(x_{2n}, x_{2n+2})}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2})}$$

$\leq d(x_{2n}, x_{2n+1})$.

Continuing in this way we obtain

$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}) = \ldots \leq d(x_0, x_1)$. So we have a monotonic sequence of positive real numbers.

Let the monotonic sequence converges to a real number l. Since $\{x_n\}$ has convergent subsequence $\{x_{n_k}\}$ in X which converges to some $z \in X$, therefore

$$\lim_{n \to \infty} x_{2n_k} = z.$$

Then we show that z is a common fixed point of SP and TP. If it is not possible we assume that $z \neq SPz$.

Now

$$\begin{aligned} d(z,SPz) &= d(\lim x_{2n_k}, SP\, lim\, x_{2n_k}) \\ &= \lim d(x_{2n_k}, SPx_{2n_k}) \\ &= \lim d(x_{2n_k}, x_{2n_{k+1}}) \\ &= \lim d(x_{2n_{k+1}}, x_{2n_{k+2}}) \\ &= \lim d(SPx_{2n_k}, TPSPx_{2n_k}) \\ &= d(\lim SPx_{2n_k}, \lim TPSPx_{2n_k}) \\ &= d(SPz, TPSPz) \\ &\leq \frac{d(z, SPz)\, d(SPz, TPSPz) + d(SPz, SPz)\, d(z, TPSPz)}{d(z, sPz) + d(z, TP\, SPz)} \end{aligned}$$

i.e. $d(z, SPz) \leq d(z, SPz)$

which is a contradiction. Therefore $z = SPz$, so that z is a fixed point of SP.

In the same way let $z \neq TPz$, we get

$$d(z, TPz) = d(SPz, TPSPz). \leq d(z, SPz) = 0$$

a contradiction. Hence z is a fixed point of TP also. Thus

$$SPz = z = TPz.$$

Now we shall show that z is a common fixed point of S,P and T. Using (i) we have

$$\begin{aligned} d(Pz, z) &= d(PSPz, TPz) \\ &< d(SPPz, TPz) \\ &\leq \frac{d(Pz, Pz)\, d(z, z) + d(z, Pz)\, d(Pz, z)}{d(Pz, Pz) + d(Pz, z)} \end{aligned}$$

i.e., $$d(Pz, z) \leq d(Pz, z).$$
This implies that $Pz = z$. Hence by (i) we have $Sz = z\ Tz$. Similarly using condition (ii) we have
$$Pz = z = Sz = Tz.$$

Therefore z is a common fixed of S, T and P. For the uniqueness of fixed point, let z and z' be the two common fixed points of S, P and T. Then
$$d(z, z') = d(SPz, TPz')$$
$$\leq \frac{d(z, SPz)\, d(z', TPz') + d(z', SPz) d(z, SPz')}{d(z, SPz) + d(z, SPz')}$$
i.e. $d(z, z') \leq d(z, z')$,

which is a contradiction, therefore z is unique common fixed point of S,P and T.

This completes the proof of the theorem

REFERENCES

[1] Bhagwat and Singh, Fixed point of a pair of mapping, *Indian J. Pure. Appl. Math.* **17**(8), (1986), 994-997.

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