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(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

AN EXPANSION MAPPING THEOREM

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In a recent paper [1], Daffer and Kaneko proved the following fixed point theorem :

Theorem ([4], Theorem 3.) *Let (X, d) be a complete metric space. Let f be a surjective selfmap of X and g and injective selfmap of X which satisfy the following condition : there exists a number $q > 1$ such that*

(1) $$d(fx, fy) \geq qd(gx, gy)$$

for each $x, y \in X$. If f and g commute, then there exists a unique common fixed point of f and g.

We shall first construct and example to show that the condition of commutativity cannot be removed. Let $X = [0, 1]$ with the usual metric, $f(x) = 1 - x, g(x) = 1 - x/2$. Then f is surjective, g is injective, $d(fx, fy) = |x - y| \geq qd(gx, gy)$ for $1 < q < 2$ and (1) is satisfied. However, for each $x \in X, fgx = \varepsilon x$ and $gfx = 1 - \varepsilon + \varepsilon x$, so f and g do not commute. Also they do not have a common fixed point.

Consequently, if one removes the hypothesis of commutativity, then some other condition must be added, in order to obtain a common fixed point.

Let f and g be two selfmaps of a complete metric space (X, d). f and g are said to be compatible if, whenever $\{x_n\}$ is a sequence of points in X such that $\lim fx_n = \lim gx_n = t$, then $\lim d(fgx_n, gfx_n) = 0$.

The following result is an extension of the above Theorem to compatible maps :

Theorem. Let (X, d) be a complete metric space, f and g compatible selfmaps of X satisfying conditions (1) and $g(X) \subseteq f(X)$, f continuous. Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $g(X) \subseteq f(X)$, choose x_1 such that $fx_1 = gx_0$. In general, choose x_{n+1} such that $fx_{n+1} = gx_n$. Then, from (1),

$$d(gx_n, gx_{n+1}) \leq d(fx_n, fx_{n+1})/q = d(gx_{n-1}, gx_n)/q,$$

which implies that $\{gx_n\}$ is Cauchy, hence convergent. Call the limit z. Then $\lim gx_n = \lim fx_n = z$. Since f is continuous. $\lim fx_n = fz$. Since f and g compatible, $\lim d(gfx_n, fgx_n) = 0$, which, since $\lim fgx_n = fz$, implies that $\lim gfx_n = fz$.

From (1),

$$d(gfx_n, gx_n) \le d(f^2x_n, fx_n)/q.$$

Taking the limit as $n \to \infty$ yields

$$d(fz, z) \le d(fz, z)/q,$$

which implies that $z = fz$.

Again from (1),

$$d(gx_n, gz) \le d(fx_n, fz)/q.$$

Taking the limit as $n \to \infty$ gives $d(z, gz) \le d(z, fz)/q = 0$ and $z = gz$.

Suppose that w is also a common fixed point of f and g. Then

$$d(z, w) = d(gz, gw) \le d(fz, fw)/q = d(z, w)/q,$$

which implies that $z = w$, and the common fixed point is unique.

REFERENCE

[1] Peter Z. Daffer and Hideaki Kaneko, On expansive mappings, *Math. Japonica* **37** (1992), 733-735.