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(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

A DIFFERENCE FORMULA AND SOME LINEAR GENERATING RELATIONS FOR THE DISCRETE VARIABLE ANALOGUE OF MODIFIED JACOBI POLYNOMIALS

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ABSTRACT

In the present paper, we have defined a complex variable discrete analogue of Modified Jacobi polynomials, viz. $S^{\alpha}_{m+n}(z, \lambda, \mu, w)$ by using the theory of finite differences, by means of a $(m+n)$th partial difference formula. In this paper we have established the difference formula, addition and multiplication theorems, linear generating relations and some other interesting results.

1. Definition.

We define our polynomial set $S^{\alpha}_{m+n}(z, \lambda, \mu, w)$ by means of a $(m+n)$th partial difference formula :

$$S^{\alpha}_{m+n}(z, \lambda, \mu, w) = \frac{(1+\mu w)^{z/w}}{(m+n)\,!(z-\lambda w)^{[\alpha w]}} \underset{w}{\overset{n}{\Delta}} x \underset{w}{\overset{m}{\Delta}} iy [(z-\lambda w)^{\overline{[\alpha+m+nw]}} . (1+\mu w)^{-z/w}], \qquad \dots (1.1)$$

wherein $\underset{w}{\Delta} x$ and $\underset{w}{\Delta} iy$ are partial difference quotients, which is valid for discrete values of the variables at equal intervals viz. $(x, x-w, \dots, x-nw)$ and $iy, (iy-w, \dots, iy-mw)$. As $w \to 0$, the variables become continuous. In the limiting case ($w \to 0$ with $\mu = 1$) our polynomial set takes the form of the extended Laguere polynomial $L^{(\alpha)}_{N}(z)$, where $z = (x+iy)$, α is a non-negative integer, μ is a positive real numbber and λ is any complex number say $\lambda_1 + i\lambda_2$ such that $Re(\lambda w)$ is a natural number. We shall be using the following formulae:

$$\text{(i)} \ \underset{w}{\Delta x} f(x) = \frac{1}{w}[f(x+w) - f(x)] \Rightarrow \lim_{w \to 0} \underset{w}{\Delta x} f(x) = \frac{d}{dx} f(x) \qquad \dots (1.2)$$

$$\text{(ii)} \ x^{[\alpha w]} = x(x-w)(x-2w) \dots (x-\alpha w+w) \qquad \dots (1.3)$$

(iii) $\underset{w}{\overset{n}{\Delta}}x\,(u_x v_x) = \sum_{r=0}^{n} {}^nC_r \underset{w}{\overset{n-r}{\Delta}}(u_{x+rw}) \underset{w}{\overset{r}{\Delta}}x\,(v_x)$ ... (1.4)

(iv) $\underset{w}{\overset{r}{\Delta}} x\, x^{[\alpha w]} = \frac{r\,!}{(\alpha - r)\,!} x^{[\alpha - rw]}$ ... (1.5)

(v) $\underset{w}{\overset{r}{\Delta}} x\,(1+bw)^{-x/w} = (-b)^r (1+bw)^{-\frac{x}{w}-r}$ ... (1.6)

(vi) $(x - \lambda w + rw)^{[\alpha + r]} = w^r \left(\frac{x}{w} - \lambda + 1\right)_r (x - \lambda w)^{[\alpha w]}$ ... (1.7)

(vii) $\underset{w}{\overset{n}{\Delta}}x \left(\frac{x}{w}\right)_R = \frac{(R-n+1)}{w^n}\left(-\frac{x}{w}+n\right)_{R-n}$ ... (1.8)

The explicit form of our polynomial set comes out to be

$$S^{\alpha}_{m+n}(z, \lambda, \mu, w) = \frac{(1+\alpha)_{(m+n}}{(m+n)\,!} F_1\left[\frac{z}{w} - \lambda + 1, -m, -n; 1+\alpha; \frac{\mu w}{1+\mu w}\right] \quad \text{... (1.9)}$$

$$= \frac{(1+\alpha)_{m+n}}{(m+n)\,!} 2F_1\left[\frac{z}{w} - \lambda + 1, -m-n; 1+\alpha; \frac{\mu w}{1+\mu w}\right] \quad \text{... (1.10)}$$

$$= P_{m+n}^{(\alpha, \frac{z}{w} - \lambda - \alpha - m - n)}\left(\frac{1-\mu w}{1+\mu w}\right) \quad (1.11)$$

where $P_N^{\alpha_1, \beta_1 - N}(w)$ are the modified Jacobi polynomials.

As $w \to 0$ and $\mu = 1$, we have

$$\lim_{w \to 0} S^{\alpha}_{m+n}(z, \lambda, l, w) = \frac{(1+\alpha)_{m+n}}{(m+n)\,!} 1F_1\,[-m-n; 1+\alpha; z] \quad \text{... (1.12)}$$

$$= L^{(\alpha)}_{m+n}(z), \quad \text{... (1.13)}$$

where we name $L_N^{(\alpha)}(z) \equiv L^{(\alpha)}_{m+n}(z)$ as the extended Laguerre poly nomials.

If, however, $m = 0$ and simultaneously, the imaginary part of z vanishes, we arrive at the set of classical Laguerre polynomials $L_n^{(\alpha)}(x)$.

2. Difference Formula :

$$\underset{w}{\overset{p}{\Delta}} x \underset{w}{\overset{q}{\Delta}} iy\, S^{\alpha}_{m+n}(z, \lambda, \mu, w)$$

$$= \frac{(-\mu)^{p+q}}{(1+\mu w)^{p+q}} S^{\alpha+p+q}_{m+n-p-q}(z+p+q, \lambda, \mu, w) \quad \text{... (2.1)}$$

Proof. We have the left hand side of (2.1)

$$=\frac{(1+\alpha)_{m+n}}{(m+n)!}\underset{w}{\overset{p}{\Delta x}}\underset{w}{\overset{q}{\Delta iy}}\sum_{k=0}^{m+n}\frac{(-m-n)_k}{k!\,(1\,\alpha)_k}\left(\frac{x+iy}{w}-\lambda+1\right)_k\left(\frac{\mu w}{1+\mu w}\right)^k$$

$$=\frac{(1+\alpha)_{m+n}}{(m+n)!}\sum_{k=p+q}^{m+n}\frac{(-m-n)_k\left(\frac{x+iy}{w}-\lambda+1+p+q\right)_{k-p-q}\left(\frac{\mu w}{1+\mu w}\right)^k}{(k-p-q)!\,(1+\alpha)_k\,w^{p+q}}$$

$$=\frac{(1+\alpha)_{m+n}(-m-n)^{p+q}\mu^{p+q}}{(m+n)!\,(1+\alpha)^{p+q}(1+\mu w)^{p+q}}$$

$$\sum_{k=0}^{m+n-p-q}\cdot\frac{(-m-n+p+q)_k\left(\frac{x+iy}{w}-\lambda+1+p+q)_k\right(\frac{\mu w}{1+\mu w}\right)^k}{k!\,(1+\alpha+p+q)_k}$$

$$=\frac{(1+\alpha+p+q)_{m+n-p-q}}{(m+n-p-q)!}\left(\frac{-\mu}{1+\mu w}\right)^{p+q}{}_2F_1[-m-n+p+q,\ \frac{z}{w}-\lambda+1+p+q;1+\alpha+p+q;\frac{\mu w}{1+\mu w}]$$

Hence the result.

3. Addition and Multiplication Theorems.

An appeal to the finite difference analogue of the Taylor's theorems, *viz.*

$$f(x+y)=\sum_{p=0}^{\infty}\frac{y^{[pw]}\left\{\underset{w}{\overset{p}{\Delta x}}f(x)\right\}}{p!} \qquad \dots (3.1)$$

and
$$f(xz)=\sum_{p=0}^{\infty}\frac{\{x(z-1)\}^{[pw]}\left\{\underset{w}{\overset{p}{\Delta x}}f(x)\right\}}{p!}, \qquad \dots (3.2)$$

the latter being obtained by putting $y=x(z-1)$ in (3.1), we arrive at the following addition and multiplication theorems for our polynomial set with $m=0$ and simultaneously making the imaginary part of z vanish :

$$S_n^{\alpha}(x+x_1,\lambda,\mu,w)=\sum_{p=0}^{\infty}\frac{x_1^{[pw]}\left(\frac{-\mu}{1+\mu w}\right)^p S_{n-p}^{\alpha+p}(x+p,\lambda,\mu,w)}{p!} \qquad \dots (3.3)$$

$$S_n^{\alpha}(xx_2,\lambda,\mu,w)=\sum_{p=0}^{\infty}\frac{\{x(x_2-1)\}^{[pw]}S_{n-p}^{\alpha+p}(x+p,\lambda,\mu,w)}{p!} \qquad \dots (3.4)$$

4. Linear generating relations .

By using the usual series manipulation method, the following five linear generating relations can be established :

$$\text{(a)}\quad \sum_{N=0}^{\infty} \frac{S_N^{\alpha}(z,\lambda,\mu,w)\,[(\gamma_p)]_N\, t^N}{[(\delta_q)]_N}$$

$$= F_{q\;\;:0,1}^{p+1:0,1}\left[\begin{matrix}[(\gamma_p);1,1],[1+\alpha;1,1]:-;[\frac{z}{w}-\lambda+1;1] \\ [(\delta_q);1,1]:-;[1+\alpha;1]\end{matrix}\; t, \frac{-\mu w t}{1+\mu w}\right] \quad \text{... (4.1)}$$

$$\text{(b)}\quad \sum_{m,n=0}^{\infty} \frac{S_{m+n}^{\alpha}(z,\lambda,\mu,w)(m+n)!\, t^m T^n}{(1+\alpha)_{m+n}}$$

$$= (1-t)^{-1}(1-T)^{-1} F_1[\frac{z}{w}-\lambda+1;1,1;+\alpha;\frac{-\mu w t}{(1+\mu w)(1-t)}, \frac{-\mu w T}{(1+\mu w)(1-T)}] \quad \text{... (4.2)}$$

$$\text{(c)}\quad \sum_{m,n=0}^{\infty} \frac{S_{m+n}^{\alpha}(z,\lambda,\mu,w)\,t^m T^n}{m!\,n!\,(1+\alpha)_{m+n}}$$

$$= \sum_{r,s=0}^{\infty} \frac{(\frac{z}{2}-\lambda+1)_{r+s}\left(\frac{-\mu w}{1+\mu w}\right)^{r+s} t^r T^s}{r!\,s!\,(1)_{r+s}\,(1+\alpha)_{r+s}} F_1(-;,-;1+r+s;t;T] \quad \text{... (4.3)}$$

$$\text{(d)}\quad \sum_{m,n=0}^{\infty} \frac{(m+n)!\,S_{m+n}^{\alpha}(z,\lambda,\mu,w)}{m!\,n!}\left(\frac{t}{1+\alpha}\right)^{m+n}$$

$$= (1-\frac{2t}{1+\alpha})^{\frac{z}{w}-\lambda-\alpha}\left[1-\frac{2t}{(1+\mu w)(1+\alpha)}\right]^{-\left(\frac{z}{w}-\lambda+1\right)} \quad \text{... (4.4)}$$

(e) Using the auxiliary variable v, we have

$$\sum_{N=0}^{\infty} \frac{(N+k)!\,S_{N+k}^{\alpha}(z,\lambda,\mu,w)T^N}{N!\,k!}$$

$$= (1-T)^{(-1-\alpha-k)}(1+\mu w)^{\frac{z}{w}-\lambda+1}(1+\frac{\mu w}{1-T})^{-(z/w-\lambda+1)} S_k^{\alpha}\left(\frac{z}{1-T},\lambda,\mu,\frac{w}{1-T}\right) \quad \text{... (4.5)}$$

The following interesting special cases of these linear generating relations may be listed as follows :

$$\text{(i)} \sum_{N=0}^{\infty} S_N^{\alpha}(z, \lambda, \mu, w)\, t^N = (1-t)^{-1-\alpha}\left(1+\frac{\mu w t}{(1+\mu w)(1-t)}\right)^{-\left(\frac{z}{w}-\lambda+1\right)} \quad \text{... (4.6)}$$

As $w \to 0$ and $\mu = l$, $m = 0$ and simultaneously imaginary part in z vanishes, (4.6) converts into a known linear generating relation for the Laguerre polynomials.

$$\text{(ii)} \quad \sum_{m,n=0}^{\infty} \frac{(m+n)!\, L_{m+n}^{(\alpha)}(z)\, t^m T^n}{(1+\alpha)_{m+n}}$$

$$= (1-t)^{-1}(1-T)^{-1}\, \phi_2\left[1, 1; 1+\alpha; \frac{-zt}{1-t}, \frac{-zt}{1-T}\right] \quad \text{... (4.7)}$$

$$\text{(iii)} \quad \sum_{m,n=0}^{\infty} \frac{L_{m+n}^{(\alpha)}(z)\, t^m T^n}{m!\, n!\, (1+\alpha)_{m+n}} = \sum_{r,s=0}^{\infty} \frac{(-zt)^r (-zT)^s}{r!\, s!\, (1)_{r+s}\, (1+\alpha)_{r+s}}$$

$$. F_1[-; -, -, 1+r+s; t, T] \quad \text{... (4.8)}$$

$$\text{(iv)} \quad \sum_{m,n=0}^{\infty} \frac{(m+n)!\, L_{m+n}^{(\alpha)}(z)}{m!\, n!}\left(\frac{t}{1+\alpha}\right)^{m+n}$$

$$= \left(1-\frac{2t}{1+\alpha}\right)^{-1-\alpha} e^{-(2zt/(1+\alpha-2t))} \quad \text{... (4.9)}$$

$$\text{(v)} \quad \sum_{N=0}^{\infty} \frac{(N+k)!\, L_{N+k}^{(\alpha)}(z)\, T^N}{N!\, k!}$$

$$= (1-T)^{-1-\alpha-k} \exp(-zt/(1-T))\, L_k^{(\alpha)}(z/(1-T)) \quad \text{... (4.10)}$$

5. Some other interesting results :

If $N = m+n$ and $M = m_1 + n_1$ and we use the series manipulation method, we arrive at the following result :

$$\text{(a)} \quad \sum_{N=0}^{\infty} \frac{S_N^{\alpha}(z, \lambda, \mu, w)\left(\frac{t}{2}\right)_N [(\gamma_p)]_N\, w^N}{[(\delta_q)]_N}$$

$$= F_{q\ :0;\,1}^{p+2:0;\,1}\left[\begin{matrix} [(\gamma_p); 1, 1], [t/w; 1, 1], [1+\alpha; 1, 1]:-; \\ [(\delta_q); 1, 1] \qquad :-; \end{matrix}\right.$$

$$\left.\begin{matrix} [\frac{z}{w}-\lambda+1; 1] \\ [1+\alpha; 1] \end{matrix}\; w, \frac{-\mu w^2}{1+\mu w}\right] \quad \text{... (5.1)}$$

The rest of the following three results, *viz.*

(b) $S_N^{\alpha}(z, \lambda, \mu, w) = \sum_{k=0}^{N} (\alpha - \beta)_k S_{N-k}^{\beta}(z, \lambda, \mu, w)$ ... (5.2)

(c) $S_N^{\alpha+\beta+1}(z_1 + z_2, \lambda_1 + \lambda_2 - 1, \mu, w)$

$$= \sum_{k=0}^{N} S_{N-k}^{\alpha}(z_1, \lambda_1, \mu, w) S_k^{\beta}(z_2, \lambda_2, \mu, w) \quad \text{... (5.3)}$$

(d) $[\underset{w}{\Delta} x_1 \underset{w}{\Delta} iy_1 S_M^{\alpha}(z_1, \lambda_1, \mu, w)][\underset{w}{\Delta} x_2 \underset{w}{\Delta} iy_2 S_N^{\beta}(z_2, \lambda_2, \mu, w)]$

$$= \frac{\mu^4}{(1+\mu)^4} \sum_{k=0}^{N-2} \frac{(4)_k (1+\mu w)^{-k}}{k!} S_{M-2}^{\alpha}(z_1, \lambda_1, \mu, w)$$

$$S_{N-k-2}^{\beta}(z_2, \lambda_2, \mu, w), \; M \geq 2, \; N \geq 2 \quad \text{... (5.4)}$$

have, however been established below in brief.

(b) From equation (4.6), we have

$$\sum_{N=0}^{\infty} S_N^{\alpha}(z, \lambda, \mu, w) t^N = (1-t)^{\beta-\alpha} [(1-t)^{-1-\beta} (1 + \frac{\mu w t}{(1+\mu w)(1-t)})^{z/w - \lambda + 1}]$$

$$= (1-t)^{\beta-\alpha} \sum_{N=0}^{\infty} S_N^{\beta}(z, \lambda, \mu, w) t^N$$

$$= \sum_{N=0}^{\infty} \sum_{k=0}^{N} \frac{(\alpha-\beta)_k S_{N-k}^{\beta}(z, \lambda, \mu, w) t^N}{k!}$$

⇒(5.2)

(c) Using (4.6) to the following identity

$$(1-t)^{-2-\alpha-\beta} \left(1 + \frac{\mu w t}{(1+\mu w)(1-t)}\right)^{-\left(\frac{z_1+z_2}{2} - \overline{\lambda_1 + \lambda_2 - 1} + 1\right)}$$

$$= (1-t)^{-1-\beta} \left(1 + \frac{\mu w t}{(1+\mu w)(1-t)}\right)^{-\left(\frac{z_2}{w} - \lambda_2 + 1\right)} (1-t)^{-1-\alpha}$$

$$\cdot \left(1 + \frac{\mu w t}{(1+\mu w)(1-t)}\right)^{-\left(\frac{z_1}{w} - \lambda_1 + 1\right)},$$

we get

$$\sum_{N=0}^{\infty} S_N^{\alpha \cdot \cdot 1}(z_1 + z_2, \lambda_1 + \lambda_2 - 1, \mu, w) t^N$$

$$= \sum_{k=0}^{\infty} S_k^{\beta}(z_2, \lambda_2, \mu, w)t^k \sum_{N=0}^{\infty} S_N^{\alpha}(z_1, \lambda_1, \mu, w)t^N$$

$$= \sum_{N=0}^{\infty} \sum_{k=0}^{N} S_{N-k}^{\alpha}(z_1, \lambda_1, \mu, w)\, S_k^{\beta}(z_2, \lambda_2, \mu, w)t^N,$$

$\Rightarrow$(5.3)

(d) Using (4.6) and

$$\sum_{n=0}^{\infty} \sum_{k_1, \ldots, k_r = 0}^{m_1 k_1 + \ldots + m_r k_r < n} B(k_1, \ldots, k_r; n + m_1 k_1 + \ldots + m_r k_r),$$

$$= \sum_{n=0}^{\infty} \sum_{k_1, \ldots, k_r = 0}^{\infty} B(k_1, \ldots k_r; n + m_1 k_1 + \ldots m_r k_r),$$

where $m_1, m_2 \ldots, m_r$ are positive integers ($r \geq 1$), we have

$$\sum_{M,N=0}^{\infty} \underset{w}{\Delta} x_1 \underset{w}{\Delta} iy_1 S_M^{\alpha}(z_1, \lambda_1, \mu, w)][\underset{w}{\Delta} x_2 \underset{w}{\Delta} iy_2 S_N^{\beta}(z_2, \lambda, \mu, w)] \,.\, t^{M+N}$$

$$= (1-t)^{-2-\alpha-\beta} \left\{\underset{w}{\Delta} x_1 \underset{w}{\Delta} iy_1 \left(1 + \frac{\mu w t}{(1+\mu w)(1-t)}\right)^{-\left(z_1/w - \lambda_1 + 1\right)}\right\}$$

$$\left\{\underset{w}{\Delta} x_2 \underset{w}{\Delta} iy_2 \left(1 + \frac{\mu w t}{(1+\mu w)(1-t)}\right)^{-\left(z_2/w - \lambda_2 + 1\right)}\right\}$$

$$= \frac{\mu^4 t^4}{(1+\mu w - t)^4} \left\{1 + \frac{\mu w t}{(1+\mu w)(1-t)}\right\}^{-\left(\frac{z_1 + z_2}{w} - \lambda_1 - \lambda_2 + 2\right)}$$

$$= \frac{\mu^4}{(1+\mu w)^4} \sum_{M,N,k=0} \frac{(4)_k (1+\mu w)^{-k}}{k\,!} S_M^{\alpha}(z_1, \lambda_1, \mu, w)\, S_N^{B}(z_2, \lambda_2, \mu, w) \,.\, t^{M+N+k+4}$$

$$= \frac{\mu^4}{(1+\mu w)^4} \sum_{M=2}^{\infty} \sum_{N=2}^{\infty} \sum_{k=0}^{N-2} \frac{(4)_k (1+\mu w)^{-k}}{k\,!} S_{M-2}^{\alpha}(z_1, \lambda_1, \mu, w) \,.\, S_{N-k-2}^{\beta}(z_2, \lambda_2, \mu, w)\, t^{M+N},$$

$\Rightarrow$ (5.4)

In (5.4), roles of M and N may however be interchanged.

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