Jñānābha, Vol. 23, 1993

(Dedicated to Professor S. A. Naimpally on his 62th Birthday)

ON A CLASS OF FUNCTIONS CONNECTED WITH THE HYPER-BESSEL FUNCTIONS

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(Received : August 18, 1992)

ABSTRACT

In this paper, we introduce a class of functions of an analogous nature to the hyper-bessel functions treated by P. Delerure [4]. We establish some of their main properties: generating function, recurrence relations, differential equation and certain integral representations.

1. Introduction

The hyper-Bessel function $J^{(n)}_{\lambda_1,\ldots,\lambda_n}(x)$ which generalizes the ordinary Bessel function with respect to the number of indices, is defined by the formula

$$J^{(n)}_{\lambda_1,\ldots,\lambda_n} = \frac{\left(\frac{x}{n+1}\right)^{\lambda_1+\ldots+\lambda_n}}{\Gamma(\lambda_1+1)\ldots\Gamma(\lambda_n+1)}\ {}_0F_n\left[\lambda_1+1,\ldots,\lambda_n+1;-\left(\frac{x}{n+1}\right)^{n+1}\right]. \quad \ldots (1.1)$$

It was earlier investigated by Delerue [4].

For n=1, (1.1) reduces to the usual Bessel function $J_\lambda(x)$. The case n=2, giving rise to the so-called Bessel-function of the third order, was initially considered and studied in a series of works by P. Humbert ([10], [11], [12]) and later on by other authors (R.S. Varma [15], A.K. Agarwal [1], R.P. Agarwal [2]). More recently, I. Dimovski and V. Kiryakova, among other investigators, have obtained interesting results by using the hyper-Bessel functions $J^{(n)}_{\lambda_1, \dots, \lambda_n}(x)$ in relation with the general Bessel- type differential operator

$$B = t^{\alpha_0} \frac{d}{dt}\, t^{\alpha_1} \frac{d}{dt} \dots t^{\alpha_{n-1}} \frac{d}{dt}\, t^{\alpha_n}$$

(see, for example, [5], [13], [14]), which appears in important differential equations of mathematical physics (in particular, the second order singular differential operator

$$B = x^{-2}\,(x \frac{d}{dx} + \nu)\,(x \frac{d}{dx} - \nu)$$

occurring, e.g., in the heat equation and in some partial differential equations of degenerated or mixed type).

In the present paper we introduce a class of functions of an analogous nature to (1.1) and investigate some of their basic properties. For such functions, we write

$$C^{(n)}_{\lambda_1, \dots, \lambda_n}(x)$$

and they will be called "Bessel-Clifford functions of order n". For the sake of simplicity, we sometimes shall write $C_{\lambda_1, \dots, \lambda_n}(x)$ instead of $C^{(n)}_{\lambda_1, \dots, \lambda_n}(x)$.

The case n=1 leads to the Bessel-Clifford funtion $C^{(x)}_{\lambda}$ studied in detail by N. Hayek [6]. The importance of this funtion lies mainly in the fact that it substitutes $J_{\lambda}(x)$ advantageously in a great deal of theoretical and practical contexts (see [6] and references therein).

The functions $C_{\lambda_1, \lambda_2}(x)$ (corresoponding to n=2) were initially considered and treated by N. Hayek [7]. Later on, other several important properties of these functions were also analyzed by N. Hayek and V. Hernandez ([8], [9]).

Following a similar approach as that followed by Delerue for the hyper-Bessel functions [4], we make here a systematic study of the functions $C^{(n)}_{\lambda_1, \dots, \lambda_n}(x)$. We otbatin their generating function, recurrence formulas, the relation with the hyper-Bessel functions, and the differential equation of order $n+1$ satisfied by them.

Furthermore, some integral representations for $C^{(n)}_{\lambda_1, \dots, \lambda_n}(x)$ are also established.

On the other hand, it is necessary to point out that the kind of functions here introduced, have a simpler analytical structure than the Delerue hyper-Bessel functions. Therefore, we feel that the fucntions

$$C^{(n)}_{\lambda_1, \dots, \lambda_n}(x)$$

can be usefully exploited in many theoretical and applicative areas in which Delerue's functions and related ones appear.

In this paper, all the parameters are taken to be real unless otherwise stated.

2. The Function $C^{(n)}_{\lambda_1, \lambda_2, ..., \lambda_n}(x)$

The Bessel-Clifford function $C_\lambda(x)$ has been studied in a series of works by Hayek (see mainly [6]).

First, we recall that this function, whose manifold properties can be seen in [6], possesses the series representation

$$C_\lambda(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{\Gamma(\lambda + r + 1)\, r\,!} \tag{2.1}$$

and admits $e^{u - x/u}$ as a generating function.

It is the principal solution of the linear differential equation $x y'' + (\lambda + 1) y' + y = 0$, and is related with the usual Bessel fucntion by means of the equality :

$$C_\lambda(x) = x^{-\lambda/2} J_\lambda(2\sqrt{x}). \tag{2.2}$$

This function may be generalized with respect to the number of indices yielding

$$C^{(n)}_{\lambda_1, ..., \lambda_n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{\Gamma(\lambda_1 + r + 1) \ldots \Gamma(\lambda_n + r + 1)\, r\,!} \tag{2.3}$$

From this series representation, it is evident that

$$C^{(n)}_{\lambda_1, ..., \lambda_n}(x) = \frac{1}{\Gamma(\lambda_1 + 1) \ldots \Gamma(\lambda_n + 1)}\, {}_0F_n(\lambda_1 + 1, ..., \lambda_n + 1; -x) \tag{2.4}$$

where ${}_0F_n(\lambda_1 + 1, ..., \lambda_n + 1; -x)$ is the well-known hypergeometric function.

Henceforth, $C^{(n)}_{\lambda_1, ..., \lambda_n}(x)$ will be called "the Bessel-Clifford function of the order n and indices $\lambda_1, ..., \lambda_n$".

From (2.3) it follows that $C^{(n)}_{\lambda_1, ..., \lambda_n}(x)$ is an analytic function of x for all x, and an analytic function of λ_i for all λ_i.

Since (2.3) converges absolutedly and uniformly on any bounded interval, the series can be differentiated or integrated term by term, with reppect to x or to λ_i.

From (1.1) and (2.4), we obtain

$$C^{(n)}_{\lambda_1, ..., \lambda_n}(x) = x^{-(\lambda_1 + ... + \lambda_n)/(n+1)}\, J^{(n)}_{\lambda_1, ..., \lambda_n}\left((n+1)\, x^{1/(n+1)}\right) \tag{2.5}$$

Clearly, the definition (2.4) makes sense only when the indices λ_i are not negative integers. To avoid this inconvenience, we take as definition the following generating function of n integral parameters $m_1, ..., m_n$:

$$e^{u_1 + ... + u_n - x/(u_1 ... u_n)}$$

$$= \sum_{m_1=-\infty}^{+\infty} ... \sum_{m_n=-\infty}^{+\infty} u_1^{m_1} ... u_n^{m_n} C_{m_1, ..., m_n}(x). \tag{2.6}$$

Thus, if the integers m_i are all positive, the series (2.6) leads us agian to (2.4). If some of these m_i are negative, then we can suppose that they are increasingly ordered, i.e.,

$$-m_1, ... -m_k, +m_{k+1}, ..., +m_n,$$

and after the change of variable in the generating function

$$u_1 ... u_n = x/t ,$$

we have the following relation :

$$C_{-m_1, ... -m_k, +m_{k+1}, ..., +m_n}(x) =$$

$$= (-x)^{m_1} C_{m_1, m_1 - m_2, ..., m_1 - m_k, m_1 + m_{k+1}, ..., m_1 + m_n}(x) \tag{2.7}$$

Analogously, we can express other similar formulas in which the right-hand memebrs include powers of $(-x)$ with other subindices as exponents. All of these relations allow us to define the Bessel Clifford functions of the order n, in the cases in which some indices are negative integers and the rest non integer positive numbers.

3. Recurrence Relations and Differential Formulas with Respect to the Indices

The symbolic calculus of n variables, used by Delerue [3] to study the properties of the hyper-Bessel functions, can be also employed to establish several recurrence formulas of different types for the $C^{(n)}_{\lambda_1, ..., \lambda_n}(x)$.

Let us consider the following general symbolic relation [3] :

$$x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n} ... x_n^{\alpha_n} f(x_1, x_2, ..., x_n) \underset{n}{\supset}$$

$$(-1)^{\alpha_1 + ... + \alpha_n} p_1 ... p_n \frac{\partial^{\alpha_1 + ... + \alpha_n}}{\partial_{p_1}^{\alpha_1} ... \partial_{p_n}^{\alpha_n}} \left[\frac{\varphi(p_1, ..., p_n)}{p_1 ... p_n} \right] \tag{3.1}$$

in which $x_1, ..., x_n$ correspond to the symbolic variables $p_1, ..., p_n$, respectively.

Then, starting from the function of n variables :

$$\frac{x_1^{\lambda_1} \dots x_n^{\lambda_n}}{(\Gamma(\lambda_1+1) \dots \Gamma(\lambda_n+1)} \; {}_0F_n(\lambda_1+1, \dots, \lambda_n+1; x_1 \dots x_n)$$

and, after some manipulation, we get

$$x_1^{\lambda_1} \dots x_n^{\lambda_n} C_{\lambda_1, \dots, \lambda_n}(x_1 \dots x_n) \underset{n}{\supset}$$

$$\underset{n}{\supset} \quad \frac{1}{p_1^{\lambda_1} \dots p_n^{\lambda_n}} \, e^{-1/(p_1 \dots p_n)} \tag{3.2}$$

Let us now consider the more simple case in (3.1), i.e.

$$x_1 f(x_1, \dots, x_n) \underset{n}{\supset} -p_1 \frac{\partial}{\partial p_1} \left[\frac{\varphi(p_1, \dots, p_n)}{p_1} \right]$$

then, taking into account (3.2), the following recurrence relation, holds :

$$C_{\lambda_1-1, \lambda_2, \dots, \lambda_n}(x) + x\, C_{\lambda_1+1, \dots, \lambda_n+1}(x) = \lambda_1 C_{\lambda_1, \dots, \lambda_n}(x) \, . \tag{3.3}$$

Thus, summing all the analogous relations to the (3.3), we can write :

$$C_{\lambda_1-1, \lambda_2, \dots, \lambda_n}(x) + C_{\lambda_1, \lambda_2-1, \dots, \lambda_n}(x) + \dots +$$

$$+ C_{\lambda_1, \lambda_2, \dots, \lambda_n-1}(x) + n\, x\, C_{\lambda_1+1, \dots, \lambda_n+1}(x) =$$

$$= (\lambda_1 + \dots + \lambda_n)\, C_{\lambda_1, \dots, \lambda_n}(x) \tag{3.4}$$

Similarly, by particularizing the left-hand member of the general symbolic relation (3.1) to the case $x_1 x_2 f(x_1, \dots, x_n)$,

it is also inferred :

$$C_{\lambda_1, \dots, \lambda_n}(x) = x^2 C_{\lambda_1+3, \lambda_2+3, \lambda_3+2, \dots, \lambda_n+2}(x) +$$

$$+ (\lambda_1+1)(\lambda_2+1)\, C_{\lambda_1+1, \lambda_2+1, \lambda_3, \dots \lambda_n}(x)$$

$$- (\lambda_1 + \lambda_2 + 3)\, x\, C_{\lambda_1+2, \lambda_2+2, \lambda_3+1, \dots, \lambda_n+1}(x). \tag{3.5}$$

Moreover, when the particular symbolic relation

$$x_1 \frac{\partial f}{\partial x_1} \supset -p_1 \frac{\partial \varphi}{\partial p_1}$$

is applied to (3.2), one has :

$$C^{(p)}_{\lambda_1, \lambda_2, \dots, \lambda_n}(x) = -C_{\lambda_1+1, \lambda_2+1, \dots, \lambda_{n+1}}(x). \tag{3.6}$$

Repeatedly differentiation p times, it follows :

$$C^{(p)}_{\lambda_1, ..., \lambda_n}(x) = (-1)^p \, C_{\lambda_1+p, ..., \lambda_n+p}(x). \tag{3.7}$$

From (3.3) and (3.6), it also results :

$$\lambda_1 C_{\lambda_1, ..., \lambda_n}(x) + C'_{\lambda_1, ..., \lambda_n}(x) = C_{\lambda_1-1, \lambda_2, ..., \lambda_n}(x) \; . \tag{3.8}$$

Furthermore, we can obtain :

$$\frac{d}{dx}\left[x^{\lambda_1} C_{\lambda_1, ..., \lambda_n}(x)\right] = x^{\lambda_1 - 1} C_{\lambda_1-1, \lambda_2, ..., \lambda_n}(x) \tag{3.9}$$

which, by using (3.2), may be generalized without difficulty to the case of n variables $x_1, ..., x_n$.

The sum of similar relations to (3.8) and (3.9), respectively, yield other formulas.

On the other hand, differentiating (3.8) with respect to x, and after some algebra, we have :

$$C''_{\lambda_1, ..., \lambda_n}(x) + \frac{\lambda_1+\lambda_2+1}{x} C'_{\lambda_1, ..., \lambda_n}(x) + \frac{\lambda_1 \lambda_2}{x^2} C_{\lambda_1, ..., \lambda_n}(x) = \frac{1}{x^2} C_{\lambda_1-1, \lambda_2-1, \lambda_3, ..., \lambda_n}(x) \; . \tag{3.10}$$

A new differentitaion also gives :

$$C'''_{\lambda_1, ..., \lambda_n}(x) + \frac{\lambda_1+\lambda_2+\lambda_3+1}{x} C''_{\lambda_1, ..., \lambda_n}(x) + \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3+1}{x^2} C'_{\lambda_1, ..., \lambda_n}(x) + \frac{\lambda_1 \lambda_2 \lambda_3}{x^3} C_{\lambda_1, ..., \lambda_n}(x) = \frac{1}{x^3} C_{\lambda_1-1, \lambda_2-1, \lambda_3-1, \lambda_4, ..., \lambda_n}(x) \tag{3.11}$$

Iterating the process, we obtain the form of the more general relation including the derivative of the k^{th} order of the function $C_{\lambda_1, ..., \lambda_n}(x)$.

Remark 1. For n=1 and 2, we get again recurrence relations satisfied by $C_\lambda(x)$ and $C_{\lambda_1, \lambda_2}(x)$. For example, (3.6) and (3.7) are reduced to formula (6.1) ([6]) and (3.2), (3.3) ([7]), respectively.

Remark 2. The relations (3.10), (3.11) and the subsequent ones allow to calculate through the corresponding differential equations, the function $C_{\lambda_1, ..., \lambda_n}(x)$ by means of :

$$C_{\lambda_1-1, \lambda_2-1, \lambda_3, ..., \lambda_n}(x), C_{\lambda_1-1, \lambda_2-1, \lambda_3-1, \lambda_4, ..., \lambda_n}(x),$$

etc., respectively.

Thus, for example, from (3.10) it results the differential equation :

$$x^2 y'' + (\lambda_1 + \lambda_2 + 1)\, x\, y' + \lambda_1 \lambda_2 y = = C_{\lambda_1 - 1, \lambda_2 - 1, \lambda_3, \ldots, \lambda_n}(x)$$

which can be solved by the well-known methods.

Next, we will establish for $C_{\lambda_1, \ldots, \lambda_n}(x)$ some differentiation formulas with respect to the indices.

Using a method by Delerue [4] (generalization of the Humbert's one [11]) for the hyper-Bessel functions, we start differentiating the following symbolic relation with respect to λ_1 :

$$x^{\lambda_1} \log x\, C_{\lambda_1, \ldots, \lambda_n}(x) + x^{\lambda_1} \frac{\partial}{\partial \lambda_1}\left[C_{\lambda_1, \ldots, \lambda_n}(x)\right] \supset$$
$$\supset - p^{-\lambda_1} \log p\, C_{\lambda_1, \ldots, \lambda_n}(1/p)$$

Then, by applying the composition product theorem and taking into account the relation which is deduced integrating (3.9), it is inferred :

$$\frac{\partial}{\partial \lambda_1}\left[C_{\lambda_1, \ldots, \lambda_n}(x)\right] = \gamma\, C_{\lambda_1, \ldots, \lambda_n}(x) +$$
$$+ \frac{1}{x} \int_0^x \left[\log\left(1 - \frac{y}{x}\right)\right]\left(\frac{y}{x}\right)^{\lambda_1 - 1} C_{\lambda_1 - 1, \lambda_2, \ldots, \lambda_n}(y)\, dy \qquad (3.12)$$

where γ is the Euler constant. Other similar formulas can be deduced corresponding to the remaining values of the indices.

If all the indices are equal to m, on differentiating with respect to m the symbolic relation deduced from (3.2) by putting all the λ_i equal to m, we have, after using several symbolic calculus rules and some transformations :

$$\frac{\partial}{\partial m}\left[C_{m, \ldots, m}(x)\right] = n\, \gamma\, C_{m, \ldots, m}(x) +$$
$$+ n \int_0^1 [\log(1 - t)]\, t^{m-1} C_{m-1, m, \ldots, m}(tx)\, dt \qquad (3.13)$$

A similar result is inferred when only k of the indices were equal to m.

Remark 3. Note the more simplified forms of the abvoe formulas with respect to the ones obtained by Delerue [4] for the $J_{\lambda_1, \ldots, \lambda_n}(x)$.

In particular, (3.12) it reduces for the third-order Bessel-Clifford fucntion to the formula :

$$\frac{\partial}{\partial m}\left[C_{m, n}(x)\right] =$$

$$\gamma C_{m,n}(x) + \frac{1}{x} \int_0^x \left[\log\left(1 - \frac{y}{x}\right) \right] \left(\frac{y}{x}\right)^{m-1} C_{m-1,n}(y)\, dy$$

Identical expression holds for $C_m(x)$.

4. Differential Equation

In the previous section we have concluded that the function $C_{\lambda_1,\dots,\lambda_n}(x)$ satisfied a differential equation of the form :

$$x^n \frac{d^n y}{dx^n} + \alpha_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + \alpha_{n-k} x^k \frac{d^k y}{dx^k} + \dots +$$

$$+ \alpha_{n-1} x y' + \alpha_n y = C_{\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_{n-1}}(x) \qquad (4.1)$$

The coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$, are the same as those obtained by Delerue for the $(n+1)^{th}$ order differential equation satisfied by the hyper- Bessel functions of the order n, i.e. $\alpha_n = \sigma_n$

$$\alpha_n + \alpha_{n-1} = 1 + \sigma_1 + \dots + \sigma_n ,$$

$$k!\, \alpha_{n-k} + \frac{k!}{1!} \alpha_{n-k+1} + \frac{k!}{2!} \alpha_{n-k+2} + \dots + \quad + \frac{k!}{(k-1)!} \alpha_{n-1} + \frac{k!}{k!} \alpha_n =$$

$$= k^n + k^{n-1} \sigma_1 + k^{n-2} \sigma_2 + \dots + k^{n-k} \sigma_k + \dots + k\, \sigma_{n-1} + \sigma_n \qquad (4.2)$$

In particular, $\alpha_1 = \sigma_1 + \frac{n(n-1)}{2}$. The quantities $\sigma_1, \sigma_2, \dots, \sigma_n$, denote the sum of the partial products obtained by multiplying 1,2,..,k,..n coefficients $\lambda_1, \dots, \lambda_k, \dots, \lambda_n$.

Now, differentiating (4.1) and taking into account (3.6), one deduces the following linear differential equation of the order n+1 verified by $C_{\lambda_1,\lambda_2,\dots,\lambda_n}(x)$.

$$x^{n+1} y^{(n+1)} + \left\{ \sigma_1 + \frac{n(n+1)}{2} \right\} x^n y^{(n)} + \dots + \left\{ k\, \alpha_{n-k} + \alpha_{n-k+1} \right\} x^k y^{(k)} +$$

$$+ \dots + \left\{ \alpha_{n-1} + \alpha_n \right\} x y' + x y = 0 \qquad (4.3)$$

k taking the values $n-1$, $n-2$, ..., 3, 2, 1.

Remark 4. For n=1, we obtain the known differential equation satisfied by $C_{\lambda_1,\lambda_2}(x)$ [8].

Theorem. If the indices $\lambda_1, \lambda_2, \dots, \lambda_n \notin Z$ or $\lambda_i - \lambda_j \notin Z$ ($\forall \neq j$), the differential equation (4.3) has as fundamental system of solutions, the following n+1 functions :

$$C_{\lambda_1,\lambda_2,\dots,\lambda_n}(x),\ x^{-\lambda_1} C_{-\lambda_1,\lambda_2-\lambda_1,\dots,\lambda_n-\lambda_1}(x),$$

$$x^{-\lambda_2} C_{-\lambda_2,\lambda_1-\lambda_2,\dots,\lambda_n-\lambda_2}(x), \dots, x^{-\lambda_n} C_{-\lambda_n,\lambda_1-\lambda_n,\dots,\lambda_{n-1}-\lambda_n}(x)$$

(4.4)

Proof : The $(n+1)$ linearly independent solutions (4. 4) of (4. 3), can be easily obtained by applying the well-known Frobenius method.

5. Integral Representations of $C_{\lambda_1, \lambda_2, \dots, \lambda_n}(x)$

Starting from the definition of the generalized hypergeometric function introduced by C.S. Meijer as a Mellin-Barnes type integral [13], i.e.

$$G^{m,n}_{p,q}(z) = G^{m,n}_{p,q}\left[z \left| \begin{matrix} a_1, & a_2, \dots, a_p \\ b_1, & b_2, \dots, b_q \end{matrix} \right.\right]$$

we can establish an interesting integral representation for the treated fucntions, namely :

$$C^{(n-1)}_{\lambda_1, \dots, \lambda_{n-1}}(z)$$

$$= \left[\frac{n}{(2\pi)^{n-1}}\right]^{\frac{1}{2}} \int_0^1 G^{n-1,0}_{n-1,n-1}\left[t \left| \begin{matrix} \lambda_1, \dots, \lambda_{n-1} \\ \frac{1}{n}-1, \frac{2}{n}-1, \dots, \frac{n-1}{n}-1 \end{matrix} \right.\right] \cos_n\left[n(zt)^{1/n}\right] dt$$

with $\qquad \lambda_k > \frac{k}{n} - 1, \; k = 1, 2, \dots, n-1$

and $\qquad \cos_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{nj}}{(nj)!}$

being the generalized cosine function of order $n > 1$ (see [5]).

Note that the formula (5.1) is analogous to the one obtained by Dimovski and Kiryakova [5].

Remark 5. For n=2 and $\lambda_1 = \nu > -\frac{1}{2}, \lambda_2 = 0$ the formula (5.1) reduces to the integral representation of the Poisson type for the modified Bessel-Clifford $E_\nu(x)$ [6].

Finally, from the generating function (2.6) written as a product of the two following hypergeometric functions

$${}_0F_{n-1}\,[1, 1, \dots, 1\,;\, -\sqrt{x}\; t]\,.\,{}_0F_{n-1}\,[1, 1, \dots, 1\,;\, \sqrt{x}/t]$$

we can also infer another integral representation :

$$C^{(2n-1)}_{m,(n),m,0,(n-1),0}(x) = \frac{x^{-m/2}}{2\pi i}\,.$$

$$.\int_c^{(0^+)} t^{-61/m}\, C^{(n-1)}_{0,0,\dots,0}(-\sqrt{x}\; t)\, dt \quad (m \in \mathbf{Z}) \tag{5.2}$$

wehre $\int_c^{(0^+)}$ stands for the Hankel contour integral.

This expression is closely related to the representation deduced by Agarwaf for the n-Bessel function $A_{mn}(x)$ [1].

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