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WHEN IS THE WALLMAN COMPACTIFICATION FINITE ?

By

Aristotle Hadjiantoniou and M. G. Murdeshwar
Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

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ABSTRACT

In [1], the following questions were posed as some of the open problems:

(1) Suppose that X is a T_1 space. What are the necessary and sufficient conditons that the Wallman compactification be a one-point compactification?

(2) More generally, what are the necessary and sufficient conditions that the Wallman compactification be a finite compactification with n points?

The purpose of this note is to present solutions to these problems.

Background notation and results.

Let X be a set, and $\mathcal{B}$ a filter-base on X. $\mathcal{B}$ is said to be free if the intersection of all members of $\mathcal{B}$ is empty.

If X is a topological space, a filter-base $\mathcal{B}$ is called a *closed base* if each member of $\mathcal{B}$ is closed $\mathcal{B}$ is called. a *maximal closed base* if it is a closed base and is not properly contained in any closed base. Maximal closed bases were used by Wallman [2] to construct a T_1 compactification of an arbitrary T_1 space. It is well-known that the Wallman compactification of a Tychonoff space X coincides with the Stone-Cech compactification βX iff X is a T_4 space. Necessary and sufficient conditions are also known for βX to be of finite cardinality n, with the case $n = 1$ attracting the attention of a large number of researchers. This prompted the questions posed at the beginning of this article. At first sight, it may appear that the solution to our problem could be obtained by simply combining the conditions for coincidence of Wallman and Stone-Cech compactifications with the conditions for coincidence of Stone- Cech and Alexandroff compactifications. However, in our problem we require X to satisfy the T_1 condition only. Hence the Stone- Cech compactification need not exist. But the Wallman and Alexandroff compactifications certainly exist and are T_1 spaces. The solutions we present are surprisingly simple-looking conditions. In proving our theorem, we will use the following well-known properties of the Wallman compactification.

In what follows, X shall denote an arbitrary noncompact T_1

space, and W its Wallman compactification.

1. THEOREM. *[2]*

(1) The growth $W - X$ consists of all free maximal closed bases.

(2) If A and B are disjoint, closed subsets of X, the their W- closures are also disjoint.

Main result.

2. THEOREM. In order for $[W - X] = 1$, it is necessary and sufficient that X satisfy the condition:

Of every pair of disjoint closed sets, at least one is compact.

Proof: Necessity: Suppose that $W - X$ is the singleton $\{\infty\}$. Let A and B be closed, disjoint subsets of X. By Theorem 1, $\overline{A} \cap \overline{B} = 0$, where the bar designates closure in W. Hence, ∞ cannot belong to both $\overline{A}$ and $\overline{B}$. Say, $\infty \notin \overline{A}$. Thus, $\overline{A}$ is closed and compact in W, hence also closed and compact in X. Further, the X-closure of A coincides with its W-closure. This proves that A is compact in X.

Suffciency: Suppose that $W-X$ is not a singleton. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two distinct points of $W-X$ then $\mathcal{B}_1$ and $\mathcal{B}_2$ are free maximal closed bases. Because of their maximality, there exist $B_i \in \mathcal{B}_i$ $(i = 1, 2)$ such that $B_1 \cap B_2 = 0$ If one of them, say B_1, were compact, then the open cover $\{X - A \mid A \in \mathcal{B}\}$ must contain a finite subcover of B_1:

$$X - A_1, X - A_2, \dots, X - A_n .$$

This implies that $A_1 \cap A_2 \cap \dots \cap A_n \cap B_1 = 0$, which contradicts $\mathcal{B}_1$, being a filter-base. Therefore B_1 and B_2 are not compact.

The above theorem, as well as its proof, is easily generalized to the case of n points.

3. THEOREM. The following are equivalent:

(1) $| W - X | \leq n$.

(2) If $A_1, A_2, \dots, A_{n+1}$ are pairwise disjoint, closed subsets of X, then at least one A_i is compact.

REFERENCES

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[2] Wallman, Henry: Lattices and topological spaces, *Ann. Math.*, **39** (1938), 112-126.