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(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

ON CHARACTERIZATIONS OF FIXED POINTS

By

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ABSTRACT

Equivalent conditions are given for the existence of fixed points of self maps on metric and compact metric spaces.

1. INTRODUCTION

Jungck [1, 2] and Leader [3] employed commuting maps to obtain two criteria for the existence of fixed points of continuous maps on compact metric spaces.

The purpose of this paper is to establish some characterizations of fixed points for self maps on metric and compact metric spaces, some of which are variants of the corresponding results of Jungck [1, 2] and Leader [3].

Let f be a self map on a metric space (X, d), N the set of positive integers. For $x \in X$, define $O(x,f) = \{f^n \mid n \in N \cup \{0\}\}$. $\overline{O(x,f)}$ and $O'(x,f)$ denote the closure and derived of $O(x,f)$ respectively. For $D \subset X$, $\delta(D)$ denotes the diameter of D. f is called compact on D if there exists a compact subset Y of D containing fD. Set

$C_t = \{g \mid g : X \to X, gf = fg\}$

$CX_t = \{g \mid g : X \to X, gf = fg \text{ and } gX = X\}$

$A_t = \{g \mid : X \to X, g \cap_{n \in N} f^n X = \cap_{n \in N} f^n X$

Clearly, the identity map belongs to $CX_t \subset C_t$.

2. LEMMAS

Lemma 2.1. Let f be a continuous self map of a metric space (X, d). Assume $\{A_n\}_{n \in N}$ is a sequence of nonempty compact subsets of X such that $A_{n+1} \subset A_n$ for $n \in N$. Then $f \cap_{n \in N} A_n = \cap_{n \in N} fA_n$ and $f \cap_{n \in N} A_n$ is compact.

Proof. Since X is a metric space and $A_n \supset A_{n+1} \neq \phi$ for $n \in N$, $\cap_{n \in N} A_n$ is a nonempty compact subset by the compactness of A_n, $n \in N$. By the continuity of f it follows that $f \cap_{n \in N} A_n$ is a nonempty compact subset. Clearly $f \cap_{n \in N} A_n \subset \cap_{n \in N} fA_n$. We now prove that $f\cap_{n\in N}A_n \supset \cap_{n\in N} fA_n$. For each $x\in \cap_{n\in N} fA_n$, there exists a sequence $\{a_n\}_{n\in N}$ such that $x=fa_n$ and $a_n\in A_n$ for $n\in N$. For $m\in N$, note that $\{a_n\}_{n\geq m}\subset A_m$ and that A_m is compact, then there exists a convergent subsequence $\{a_{n_k}\}_{k\in N}$ of $\{a_n\}_{n\geq m}$ with linit, say, p. Thus $p\in A_m$ Since A_m is closed. This implies $p\in \cap_{n\in N}A_n$. By the continuity of f, we obtain $x=fa_{n_k}=fp$; i. e. , $x\in f\cap_{n\in N}A_n$. Thus $f\cap_{n\in N}A_n \supset \cap_{n\in N} fA_n$. Hence $f\cap_{n\in N}A_n = \cap_{n\in N} fA_n$. This completes the proof.

Lemma 2.2. Let f be a continuous self metric space (X, d), Y a compact subset of X with $Y\cap fY$. If $A=\cap_{n\in N} f^nY$, then A is a nonempty compact subset of X and $fA = A$.

Proof. Clearly $f^n Y$ is compact and $f^n Y\supset f^{n+1}Y$ for $n\in N$. By Lemma 2.1 it follows that $fA=\cap_{n\in N} f^{n+1}Y=A$ and that A is a nonempty compact subset of X. This completes the proof.

From Lemma 2.2 we have

Lemma 2.3. Let f be a continuous self map of a compact metric space $(X. d)$. Then $\{f^n \mid n\in N\cup\{0\}\}\subset A_t$.

Lemma 2.4. Let (X, d) be a metric space. Assume there exists a connected subset A of X containing at least two distinct points. Then A is uncountable.

Proof. See [4].

3. FIXED POINTS ON METRIC SPACES

The following result describes an interdependence between connectedness and fixed point.

Theorem 3.1. For a self map f on a metric space (X, d) the following conditions are equivalent.

(1) f has a fixed point;

(2) There exists nonempty countable connected subset A of X satisfying $fA\subset A$;

(3) There exists countable connected subset A of X satisfying $fB\subset B$ for some nonempty subset B of A;

(4) There exists $x\in X$ such that $O(x, f)$ is connected;

(5) Tere exists $x\in X$ such that $\overline{O(x,f)}$ is connected and that $O'(x,f)$ is countable.

Proof. We shall verify the following implications: (1) → (2) → (3) → (1), (1) → (4) → (2) and (1) → (5) → (3). Let (1) holds and x be a fixed point of f. Take $A=O(x,f)$. Then $O'(x,f)=\phi$, $\overline{O(x,f)}=A=\{x\}$. Hence (2), (4) and (5) holds,. Clearly (2) implies (3).

Let (3) holds. Note that A is countable and connected. By Lemma 2.4 it follows that A is a singleton. Let $A = \{a\}$ for some $a \in X$. Since there exists a nonempty subset B of A such that $fB \subset B$ of $B = A$. This implies $a = fa$. Hence **(1) holds.**

Let (4) holds. Take $A = O(x, f)$. Then $fA = O(fx, f) \subset A$ and A is countable. Hince (2) holds.

Let (5) holds. Take $A = \overline{O(x, f)}$ and $B = O(x, f)$. This B and $O'(x, f)$ are countable. So is A. Clearly $\phi \neq B \subset A$ and $fB \subset B$. Hence (3) holds, This completes the proof.

For contractive maps we have the folowing two results.

Theorem 3.2. Let f be a continuous self map of a metric space (X, d). Assume there exist $m, n \in N \cup \{0\}$ such that $f^m x \neq f^n y$ implies

$$(*) \qquad d(f^m x, f^n y) < \delta\,\overline{(O(x, f)} \cup \overline{O(y, f)})$$

Then (1) is equivalent to each of the following:

(6) There exists $x \in X$ such that $O(x, f)$ is compact;

(7) There exists $x \in X$ such that $\overline{O(x, f)}$ is compact;

(8) There exists $x \in X$ and $p \in N$ such that f^p is compact on $O(x, f)$.

Proof. We shall verify the following implications: (1) $\rightarrow$ (6) $\rightarrow$ (7) $\rightarrow$ (1) and (1) $\rightarrow$ (8) $\rightarrow$ (6). Clearly (1) implies (6) and (8).

Let (6) hold. Since $O(x, f)$ is compact and X is a metric space. $O(x, f)$ is closed. Thus $\overline{O(x, f)}$ is compact. Hence (7) holds.

Let (7) hold. Since f is continuous, $f\,\overline{O(x, f)} \subset \overline{fO(x, f)} \subset \overline{O(x, f)}$. Set $A = \cap_{n \in N} f^n\, \overline{O(x, f)}$. By Lemma 2.2 it follows that A is a nonempty compact subset of X and $fA = A$. We claim that A is a singleton. If not, then $\delta(A) > 0$. By the compactness of A there exist $u, v \in A$ such that $d(u, v) = \delta(A)$. Obviously we can find $a, b \in A$ such that $f^m a = u, f^n b = v$. Note that $\overline{O(a, f)} \cup \overline{O(b, f)} \subset \overline{A} = A$. Using (*)

$$0 < d(f^m a, f^n b) < \delta\,\overline{(O(a, f)} \cup \overline{O(b, f)}) \leq \delta(A)$$

which implies

$$\delta(A) = d(u, v) = d(f^m a, f^n b) < \delta(A)$$

which is impossible. Thus A is a singleton. Let $A = \{w\}$. Then $Fw = w$. Hence (1) holds.

Let (8) holds. Since f^p is compact on $O(x, f)$, there exists a compact subset Y of $O'(x, f)$ containing $f^p O(x, f)$. Note that $O(x, f) - Y \subset O(x, f) - f^p O(x, f) = \{x, fx, \ldots, f^{p-1}x\}$. Then $O(x, f) - Y$ is finite. This implies $O(x, f) - Y$ is a compact subset of $O(x, f)$. Clearly $O(x, f) = Y \cup (x, f) - Y)$ and $O(x, f)$ is compact. Hence (6) holds. This completes the proof.

Theorem 3.3 Let f be a continuous self map of a metric space

(X, d) satisfuing

$$d(fx, fy) < \delta\,\overline{(O(x,f) \cup O(y,f))}$$

for distinct $x, y \in X$.. Then (1), (6), (7) and (8) are equivalent.

Since the proof of Theorem 3.3 is similar to that of Theorem 3.2, so we omit the proof.

4. FIXED POINTS ON COMPACT METRIC SPACES

In this section we give eight necessary and sufficient conditions for the existence of fixed points of a continuous map on compact metric spaces.

Theorem 4.1. Let f be a continuous self map of a compact metric space (X, d). Then (1) is equivalent to each of the following:

(9) There exists $x \in X$ such that $\lim_{n \to \infty} d(f^n x, f^{n+1} x) = 0$;

(10) there exists $x \in X$ such that $\lim_{n \to \infty} d(f^n x, f^{n+1} x) = 0$;

(11) For each $g \in CX_t$, $gx \neq gy$ implies

$(*)'$ $\qquad d(gx, gy) > inf\{d(hx, hy) \mid h \in C_t\}$

(12) There exists $g \in CX_t$ such that $gx \neq gy$ implies $(*)'$ holds;

(13) For each $g \in A_t$, $gx \neq gy$ implies $(*)'$ holds

(14) There exists $g \in A_t$ such that $gx \neq gy$ implies $(*)'$ holds.

Proof. We shall verify the following implications : (1) $\to$ (9) $\to$ (10) $\to$ (1) $\to$ (11) $\to$ (12) $\to$ (1) and (1) $\to$ (13) $\to$ (14) $\to$ (1). Clearly (1) implies (9), (9) implies (10), (11) implies (12) and (13) implies (14).

Let (10) holds. Since $\lim_{n \to \infty} inf\, d(f^n x, f^{n+1}, x) = 0$, there exists a strictly increasing sequence $\{n_i\}_{i \in N} \subset N$ such that $\lim_{i \to \infty} d(f^{n_i} x, f^{n_i + 1} x) = 0$.

Note that X is a compact metric space. Without loss of generality we assume that $\lim_{i \to \infty} f^{i_n} x = a \in X$. By the continuity of f and d we have

$d(a, fa) = \lim_{i \to \infty} d(f^{n_i} x, f^{n_i + 1} x) = 0$; i. e. $a = fa$. Hence (1) holds.

Let (1) holds and w be a fixed point of f. Define t:- $X \to X$ by $tx \equiv w$ for $x \in X$. Then $t \in C_t$ and $d(tx, ty) \equiv 0$ for $x, y \in X$. if $g \in CX_t \cup A_t$ and $gx \neq gy$, then $d(gx, gy) > 0 = inf\ \{d(hx, hy) \mid h \in C_t\}$. Hence (11) and (13) hold.

Let (12) holds. By the continuity of f and compactness of X, there exists $a \in X$ such that $d(a, fa) = inf\{d(x, fx) \mid x \in X\}$. It follows from $g \in CX_t$ that there exists $x \in X$ such that $gx = a$. We assert that $a = fa$. Otherwise $a \neq fa$; *i. e.*, $gx \neq fgx = gfx$. Using $(*)'$

$d(a, fa) = d(gx, gfx) > inf\{d(hx, hfx) \mid h \in C_t\}$

$= inf\{d(hx, fhx) \mid h \in C_t\} \geq d(a, fa)$ which is a contradiction. Hence $a = fa$. Thus (1) holds.

Let (14) holds Set $A = \cap_{n \in N} f^n X$. By Lemma 2.2 it follows that A is a nonempty compact subset of X and $fA = A$. Since f is continuous and A is compact, there exists $a \in A$ such that $d(a, fa) = inf\{d(x, fx) \mid x \in A\}$. Note that $g \in A_t$. Then there exists $x \in A$ such that $gx = a$. Similarly we can prove that $a = fa$. Hence (1) holds. This completes the proof.

Note that the identity map $\in CX_t$ and that $f \in A_t$ by Lemma 2.3. Then the following result follows fuom Theorem 4.1.

Corollary 4.2. Let f be a continuous self map of a compact metric space (X, d). Then (1) is equivalent to each of the following:

(15) $x \neq y$ implies $d(x, y) > inf\{d(hx, hy) \mid h \in C_t\}$;

(16) $fx \neq fy$ imples $d(fx, fy) > inf\{d(hx, hy) \mid h \in C_t\}$

Remark. Corollary 4.2 is variants of the corresponding results of Jungck [1,2] and Leader [3].

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