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FIXED POINTS OF WEAKLY COMMUTING MAPPINGS ON MENGER SPACES

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ABSTRACT

In this note we prove fixed point theorems for four mappings on a Menger space. These results are good extensions and improvements of Stojakovic [23] and several other known results on metric and Menger spaces.

INTRODUCTION

Sehgal and Bharucha-Reid [16] initiated the study of fixed points in a subclass of probabilistic metric spaces. Subsequently, there has been a steady growth of fixed point theorems in probabilistic metric and Menger spaces (see, for instance, [1], [3], [8], [13], [14], [19]- [21], [23], [24], [27] and a good bibliography may be found in [21].In fact, fixed point theorems for certain contractive type mappings on Menger spaces with proper choice of the "norm" may be obtained from corresponding results in metric spaces (see, [8] and [14]).

On the other hand, the potential of commuting mappings (see,jungck [10]) was used by various authors to present a number of fixed point theorems in metric spaces by generalizing the spatial structure or the constraints on the mappings. Sessa [17] (see also [11]-[13], [18], [21]) relaxed the concept of commuting to that of weakly commuting mappings (cf. Def.1) and observed that commuting mappings are weakly commuting but not conversely, Recently,two less restrictive concepts than of weak commutativity, in independent formulations [11] and [25],were introduced under the names of compatible mappings and asymptotically commuting mappings.

In this paper we prove common fixed point theorems for four mappings under the condition of weak commutativity. We take only one of the mappings continuous. Our results extend and improve certain results, among others, of [2]-[7], [9], [11], [12], [18], [23], [26] and [27]. The method of proof suggests that our results are valid for asymptotically commuting or compatible mappings.

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PRELIMINARIES

A probabilistic metric space (PM_space) is an ordered pair $(X, \mathcal{F})$ consisting of a nonempty set X and a mapping $\mathcal{F}$ from $X \times X$ to L, the collection of all distribution functions. The value of t at $(u, v) \in X \times X$ is generally represented by $F_{u,v}$ or, for the sake of convenience by $F(u,v)$. The functions $F(u,v)$ are assumed to satisfy the following conditions:-

(a) $F(u, v; x) = 1$ for all $x > o, u = v$;

(b) $F(u, v, 0) = 0$ for all u, v in X ;

(c) $F(u, v) = F(v, u)$ for all u, v in X;

(d) If $F(u, v; x) = 1$, $F(v, w; y) = 1$ then $F(u, w; x + y) = 1$ for all $u, v; w$ in X and $x, y > 0$.

A Menger space is a triplet $(X, \mathcal{F}, t)$, where $(X, \mathcal{F})$ is a *PM*-space and t is t-norm [15] such that the inequality:

$F(u, w; x+y) \geq t\{F(u, v; x), F(v, w; y)\}$ holds for all $u, v, w \in X$ and all $x \geq 0, y \geq 0$.

For Topological preliminaries on a Menger space, Schweizer and Sklar [15] makes an interesting reading.

Definition 1 [17]. Two self-mappings g and h of a metric space (M,d) are weakly commuting iff

$$d(ghu, hgu) \leq d(hu, gu) \text{ for all } u \in M .$$

Defintion 2. Two self-mappings g and h of a *PM*-space X afe weakly commuting iff

$$F(ghu, hgu; x) \geq F(hu, gu; x) \text{ for all } u \in X \text{ and } x > 0.$$

Definition 3 [11]-[12]. Two Self-mappings g and h of a metric space (M, d) are compatible (also called y-asymptotically commuting [25]) iff $\lim_n d(ghu_n, hgu_n) = 0$ when $\{u_n\}$ is a sequence such that $\lim_n hu_n = \lim_n gu_n = y$ for some y in X.

Definition 4 [13]. Two self-mappings g and h of a *PM*-space X are compatible (also called y-asymptotically commuting) iff $F(ghx_n, hgx_n; x) \longrightarrow 1$ for all $x > 0$ whenever $\{x_n\}$ is a sequence in X such that $gx_n, hx_n \longrightarrow y$ for some y in X

Clearly, if g and h commute or weakly commute then they are compatible but the converse is not true (see [11], [12], [20]).

EXAMPLES

Example 1. Let $M = [1, \infty]$ with the absolute value metric d and let g and h be two mappings on M defined by $gu = 1+2u$ and $hu = 1+4u$.

Then $d(ghu, hgu) = 2 \leq 2u = d(hu, gu)$. Hence the mappings are weakly commuting. Note that g and h do not commute.

In the above example there exists no sequence $\{u_n\}$ in M for which the condition of cmpatibility (or asymptotic commutativity) may be satisfied.Thus, weakly commuting mappings mappings may be vacuously compatible (or asymptotically commuting).

Example 2. Let $X = \{p, q, r\}$ and let $\mathcal{F}$ be defined via

$$F(p, r; x) = F(r, p; x) = F(r, q; x) = F(q, r; x)$$

$$= \begin{cases} 0, \text{ if } x \leq 0, \\ \frac{1}{2}, \text{ if } 0 < x \leq 2, \\ 1, \text{ if } x > 2. \end{cases}$$

and

$$F(p, q; x) = F(q, p; x) = \begin{cases} 0, \text{ if } x \leq 0, \\ \frac{1}{2}, \text{ if } 0 < x \leq 3/2, \\ 1, \text{ if } x > 3/2. \end{cases}$$

Then $(X, \mathcal{F}, t_m)$ is a Menger space [14, Ex.3.2]. Let f,g: $X \longrightarrow X$ be such that

$f(p) = f(q) = p$, $f(r) = q$

and $g(p) = g(q) = g(r) = p$. Then it can be verified that f and g are weakly commuting but not commuting.

FIXED POINT THEOREMS

Theorem 1. Let $(X, \mathcal{F}, t)$ be a complete Menger space where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$ and mappings $h, k : X \longrightarrow X$, $f: X \longrightarrow h(X)$ and $g : X \to k(X)$ be such that f and g weakly commute with k and h respectively and one of f, g, h, k, be continuous.

Further, let for all $u, v \in X$ and $x \geq 0$

(1) $F(fu, gv: x) \geq t\{F(ku, hv; \phi(x), t\{F(fu, ku; \phi(x), F(gv, hv; (\phi(x))\}\}$

hold, where $\phi : R^+ \longrightarrow R^+$ is an increasing function such that $\lim_n \phi^n(x) = \infty$ for all $x > 0$. Then f, g, h and k have a unique common fixed point. Indeed, the pairs (f, k) and (g, h) have unique common fixed points and these two points coincide.

Proof. For $u_0 \in X$, we choose a sequence $\{u_n\}$ in X such that

$$fu_{2n} = hu_{2n+1} = y_{2n} \text{ (say), and}$$

$$gu_{2n-1} = ku_{2n} = y_{2n-1} \text{ (say).}$$

This is valid since $f: X \longrightarrow h(X)$ and $g: X \longrightarrow k(X)$

By (1), for $x > 0$

$$F(y_{2n}, y_{2n+1}; x) = F(fu_{2n}, gu_{2n+1}; x)$$

$$\geq t\left\{F(y_{2n-1}, y_{2n}; \phi(x)), t\left\{F(y_{2n}, y_{2n-1}; \phi(x)), F(y_{2n+1}, y_{2n}; \phi(x))\right\}\right\}.$$

Using the associativity of t and $t(x, x) \geq x$,

$$F(y_{2n}, y_{2n+1}; x) \geq t\left\{F(y_{2n}, y_{2n-1}; \phi(x)), F(y_{2n}, y_{2n+1}; \phi(x))\right\}$$

$$\geq t\left\{F(y_{2n}, y_{2n-1}; \phi(x)), t\left\{F(y_{2n}, y_{2n-1}; \phi^2(x)), F(y_{2n}, y_{2n+1}; \phi^2(x))\right\}\right\}$$

$$\geq t\left\{F(y_{2n}, y_{2n-1}; \phi(x)), F(y_{2n}, y_{2n+1}; \phi^2(x))\right\}$$

$$\geq \quad \ldots\ldots \quad \ldots\ldots$$

$$\geq t\left\{F(y_{2n}, y_{2n-1}; \phi(x)), F(y_{2n}, y_{2n+1}; \phi^2(x))\right\}$$

Since $F(y_{2n}, y_{2n+1}; \phi^r(x)) \to 1$ *as* $r \longrightarrow \infty$, we get

$$F(y_{2n}, y_{2n+1}; x) \geq F(y_{2n-1}, y_{2n}; \phi(x)).$$

Similarly

$$F(y_{2n+1}, y_{2n+2}; x) \geq F(y_{2n}, y_{2n+1}; \phi(x)).$$

Thus, in general

(2) $F(y_n, y_{n+1}; x) \geq F(y_{n-1}, y_n; \phi(x))$.

Now for $\varepsilon > 0$ and positive integers m, n $(m > n)$, we have with the application of (1)

$$F(y_n, y_m; \phi(\varepsilon)) \geq t\left\{F(y_n, y_{n+1}; (\phi(\varepsilon) - \varepsilon), F(y_{n+1}, y_m; \varepsilon)\right\}$$

$$\geq t\left\{F(y_0, y_1; \phi^n(\phi(\varepsilon) - \varepsilon)), F(y_{n+1}, y_m; \varepsilon)\right\}$$

$$\geq t\left\{F(y_0, y_1; \phi^n(\phi(\varepsilon) - \varepsilon)), t\left\{F(y_{n+1}, y_{n+2}; (\phi(\varepsilon) - \varepsilon)), F(y_{n+2}, y_m; \varepsilon)\right\}\right\}$$

$$\geq t\left\{f(y_0 y_1; \phi^n(\phi(\varepsilon) - \varepsilon), t\left\{F(y_0, y_1; \phi^{n+1}(\phi(\varepsilon) - \varepsilon)), F(y_{n+2}, y_m; \varepsilon)\right\}\right\}$$

$$\geq t\left\{\ldots t\left\{\ldots t\left\{\ldots F(y_0, y_1; \phi^{m-1}(\varepsilon))\right\}\right\}\right\}$$

Thus $F(y_n, y_m; \phi(\varepsilon)) \to)$ *as* $n \to \infty$.

Hence $\{y_n\}$ is a Cauchy sequence and it converges to some point z (say) in X. The sequence $Fu_{2n} = hu_{2n+1} \to z, gu_{2n+1} = ku_{2n} \to z$.

Let k be continuous. Then $kfu_{2n} \to kz$ and $k^2 u_{2n} \to kz$. Since f and g commute weakly, for $\varepsilon > 0$ *we have*

$$F(fku_{2n}, kz; \varepsilon) \geq t\left\{F(fku_{2n}, kfu_{2n}; \varepsilon/2), F(kfu_{2n}, kz; \varepsilon/2)\right\}$$

$$\geq t\left\{F(ku_{2n}, fu_{2n}; \varepsilon/2), F(kfu_{2n}, kz; \varepsilon/2)\right\}$$

$$\geq 1 - \lambda \text{ as } kfu_{2n} \to kz \text{ and } ku_{2n}, fu_{2n} \to z.$$

Thus $fku_{2n} \to kz$.

Since $\{k^2 u_{2n}\}, \{fku_{2n}\}$ converge to kz and $\{gu_{2n+1}\}, \{hu_{2n+1}\}$ converge to z, for $\varepsilon, \lambda > 0$ there exists $N = N(\varepsilon, \lambda)$ such that

(3) $F(k^2u_{2n}, fku_{2n}; \phi(\varepsilon) - \varepsilon) > 1 - \lambda, F(gu_{2n+1}, hu_{2n+1}; \varepsilon/2 > 1 - \lambda$

and $F(gu_{2n+1}, z; \phi(\varepsilon) - \varepsilon) > 1 - \lambda$ for $n \geq N$.

Now by (1),

$F(fku_{2n}, z; \phi(\varepsilon)) \geq t\{F(fku_{2n}, gu_{2n+1}; \varepsilon), F(gu_{2n+1}, z; \phi(\varepsilon) - \varepsilon)\}$

$\geq t\{t\{F(k^2u_{2n}, hu_{2n+1}; \phi(\varepsilon)), t\{F(fku_{2n}, k^2u_{2n}; \phi(\varepsilon)),$

$F(gu_{2n+1}, hu_{2n+1}; \phi(\varepsilon)), F(gu_{2n+1}, z; \phi(\varepsilon) - \varepsilon$

$\geq t\{t\{F(k^2u_{2n}, fku_{2n}; \phi(\varepsilon) - \varepsilon), F(fku_{2n}, gu_{2n+1}; \varepsilon/2),$

$F(gu_{2n+1}, hu_{2n+1}; \varepsilon/2), t\{F(fku_{2n}, k^2u_{2n}; \phi(\varepsilon)),$

$F(gu_{2n+1}, hu_{2n+1}; \phi(\varepsilon)\}\}, F(gu_{2n+1}, z; \phi(\varepsilon) - \varepsilon)\}$

$= t\{F(k^2u_{2n}, fku_{2n}; \phi(\varepsilon) - \varepsilon), F(gu_{2n+1}, hu_{2n+1}\ \varepsilon/2),$

$F(gu_{2n} + 1, z; \phi(\varepsilon) - \varepsilon)\}$

$> 1 - \lambda$ from (3).

Therefore $fku_{2n} \rightarrow z$, proving $kz = z$.

Again by (1),

$F(fz, gu_{2n+1}; \varepsilon)$

$\geq t\{F(kz, hu_{2n+1}; \phi(\varepsilon)), t\{F(fz, kz; \phi(\varepsilon)),$

$F(gu_{2n+1}, hu_{2n+1}; \phi(\varepsilon))\}\}$

$= t\{F(z, hu_{2n+1}; \phi(\varepsilon)), t\{F(fz, z; \phi(\varepsilon)), F(gu_{2n+1}, hu_{2n+1}; \phi(\varepsilon))\}\}$

$\geq t\{F(z, hu_{2n+1}; \phi(\varepsilon)), t\{F(fz, gu_{2n+1}; (\varepsilon)), F(gu_{2n+1}, z; \phi(\varepsilon) - \varepsilon),$

$F(gu_{2n+1}, hu_{2n+1}; \phi(\varepsilon))\}\}$

$> 1 - \lambda$, by (3),

This implies $fz = z = kz$. Since $f: X \rightarrow h(X)$, there exists a z' in X such that $fz = z = hz'$. So, by (1),

$F(z, gz'; x) = f(fz, gz'; x)$

$\geq t\{F(kz, hz'; \phi(x)), t\{F(fz, kz; \phi(x)), F(gz', hz'; \phi(x))\}\}$

$= t\{f(z, z; \phi(x)), t\{F(z, z; \phi(x)), F(gz', z; \phi(x))\}\}$

$= F(z, gz'; \phi(x))$, implying

$z = gz'$

Also, by the weak commutativity of g and h

$F(ghz', hgz'; x) \geq F(hz', gz'; x) = F(z, z; x) = 1,$

giving $gz = ghz' = hgz' = hz$.

Now by (1), we have

$F(z, gz; x) = F(fz, gz; x)$

$\geq t\{F(kz, hz; \phi(x)), t\{F(fz, kz; \phi(x), F(gz, hz; \phi(x))\}\}$

$= F(z, gz; \phi(x))$.

This gives $z = gz$.

Thus z is the common fixed point of f, g, h and k.

A similar proof works for the case when h is continuous instead of k.

We now suppose that f is continuous. In this case $fku_{2n} \to fz$ and also $f^2u_{2n} \to fz$. By the weak commutativity of f and k, for $\varepsilon > 0$,

$$F(kfu_{2n}; fz; \varepsilon) \geq t\{F(kfu_{2n}, fku_{2n}; \varepsilon/2), F(fku_{2n}, fz; \varepsilon/2)\}$$
$$\geq t\{F(fu_{2n}, ku_{2n}; \varepsilon/2), F(fku_{2n}, fz; \varepsilon/2)\}$$
$$> 1 - \lambda .$$

Therefore $kfu_{2n} \to fz$. Now with the application of (1) we can prove $fz = z$.

As in the previous case, one easlily shows $hz = gz = z$. Since $g:X \to k(X)$, there exists z'' in X such that $gz = z = kz''$. Using (1) again we obtain $fz'' = z$. Since f and k commute weakly, for $x > 0$, we have

$$F(fkz'', kfz''; x) \geq F(kz'', fz''; x) = F(z, z; x) = 1,$$

implying

$$z = fz = fkz'' = kfz'' = kz .$$

Thus z is a common fixed point of f, g, h and k. An analogous proof may be furnished when g is taken continuous in place of f.

The uniqueness of z again follows from (1).

Theorem 2. Let (M, d) be a complete metric space and

$h, k : M \to M, f : M \to h(M), g : M \to k(M)$

satisfy

$$d(fu, gv) \geq \psi(d(ku, hv), d(fu, ku), d(gv, hv))$$

for all $u, v \in M$, where $\psi : R^+ \to R^+$ is an increasing function such that ψ^{-1} is right upper semicontinuous, $\psi(x) < x$ and $\lim_{x} (x - \psi(x)) = \infty$ for all $x \in R^+$. If f and g weakly commute with k and h respectively and one of f, g, h, k, is continuous, then f, g, h, and k have a unique common fixed point. Indeed, the pairs (f, k) and (g, h) have unique common fixed points and these two points coincide.

Proof: We note that if (M, d) is a metric space, then induces a mapping $\mathcal{F} : M \times M \longrightarrow L$, where $\mathcal{F}(v, v) = F(u, v; x) = H(d - d(u, v))$ were H is a distribution function such that $H(x) = 0$ for $x \leq 0$ and $H(x) = 1$ for $x > 0$. Further, if the t-norm is defined by $t(x, y) = \min\{x, y\}$, then $(M, \mathcal{F}, t)$ is a Menger space. Taking $\psi^{-1} = \phi$, the proof is complete by Theorem 1.

Remark. One may improve Theorems 1-2 by replacing weak commutativity of the maps by their compatibility; and while doing so, the relevant part of the proof (cf. Th.1) is slightly simplified.

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