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(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

(1, 1 RINGS)

by

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ABSTRACT

A simple not associative (1, 1) ring, with commutators in the left nucleus, has no proper left ideals.

A left primitive (1, 1) ring, with commutators in the left nucleus and associators in the centre, is either associative or simple with a right identity element. Let A be a left ideal of (1, 1) ring R with commutators in the nucleus N. If A is maximal and nil, then A is a two-sided ideal. If A *is* minimal, then it is either a two-sided ideal, or the ideal it generates is contained in N.

1. INTRODUCTION

A (1, 1) ring R is a non-associative ring in which the following identities hold:

(1) $(x, y, z) = (x, z, y)$,

(2) $(x, y, z) + (y, z, x) + (z, x, y) = 0$,

for all x, y, z in R where the asociator $(x, y, z) = (x\,y)\,z - x\,(y\,z)$.

(R, R, R) is linear span of all associators in R .Kleinfeld [3] proved that a simple (1, 1) ring with a non-zero idempotent is associative.

We shall denote the left nucleus by L, nucleus by N and centre by U. Thus L consists of all elements l in R such that $(l, R, R) = (0)$. In (1, 1) ring R, N consists of all elements n in R such that $(R, n, R) = (0)$. By virtue of (1), $(R, R, n) = (0)$. Using (2), we have $(n, R, R) = (0)$. So $N \subseteq L$. It is well known [4] that N is an associative subring of R. U consists of all elements u in R such that $(u, R) = (0)$, where the commutator $(x, y) = xy - yx$. (R, R) is linear span of all commutators in R.

2. SIMPLE RINGS

Lemma 1. If R is simple, not associative and $(R, R) \subseteq L$, then R has no proper left ideals.

Proof. Suppose that A is a nonzero left ideal of R. Then $(R, R, A) \subseteq A$; hence from (1), we have $(R, A, R) \subseteq A$ and from (2), we have $(A, R, R) \subseteq A$. We shall show that $A = R.\,A + AR$ is a two sided ideal of R, since

$$(A + AR)\, R \subseteq AR + (AR)\, R \subseteq AR + A\,(RR) + (A, R, R) \subseteq AR + A,$$

and

$$R\,(A + AR) \subseteq RA + R\,(AR) \subseteq A + (RA)\, R + (R, A, R) \subseteq A + AR.$$

Since $A \neq (0)$ and R is simple, $R = A + AR$. Hence

$$\begin{aligned}(R, R, R) &\subseteq (A + AR, R, R) \subseteq (A, R, R) + (AR, R, R)\\ &\subseteq (A, R, R) + (RA, R, R) \subseteq (A, R, R) + (A, R, R)\\ &\subseteq (A, R, R) \subseteq A.\end{aligned}$$

Since $RA \subseteq A$, $(R, R, R) + R(R, R, R) \subseteq A$. But it is known [2] that $(R, R, R) + R\,(R, R, R)$ is a two-sided ideal of R, hence it is equal to R since R is not associative. Therefore $A = R$.

In any nonassociative ring we have

$$(xy, z) - x\,(y, z) - (x, z)\, y = (x, y, z) - (x, z, y) + (z, x, y).$$

Using (1), the above identity becomes

(3) $$(x\,y, z) - x\,(y, z) - (x, z)\, y = (z, x, y).$$

Lemma 2. Suppose R is simple, not associative and $(R, R) \subseteq N$. If T is a subset of N, then $(T, R) = (0)$.

Proof. First we show that $(T, R) + R(T, R)$ is a left ideal of R.

$$\begin{aligned}R\,((T, R) + R\,(T, R)) &= R\,(T, R) + R\,(R\,(T, R))\\ &\subseteq R\,(T, R) + (R, R)\,(T, R) + (R, R\,(T, R))\\ &\subseteq R\,(T, R) + R\,(T, R)\ (\because (T, R) \subseteq (R, R) \subseteq N)\\ &\subseteq R\,(T, R) \subseteq (T, R) + R\,(T, R).\end{aligned}$$

Therefore $(T, R) + R\,(T, R)$ is a left ideal of R.

By Lemma 1, either $(T, R) + R\,(T, R) = R$ or $(T, R) + R\,(T, R) = (0)$.

Suppose $(T, R) + R\,(T, R) = R$. Let $y \in T$ and $x, z, \in R$. Then by (3)

$$(x, z)\, y = (xy, z) - x\,(y, z) - (z, x, y).$$

$y \in T \subseteq N$ implies that $(z, x, y) = 0$. Therefore,

$$(x, z)\, y = (x\,y, z) - x\,(y, z)$$

or

$$x\,(y, z) = (x\,y, z) - (x, z)\, y.$$

Now $y \in T \subseteq N$, $(x, z) \in (R, R) \subseteq N$ and N is a subring of R.

This implies that $(x, z)\, y \in N$. Also $(xy, z) \in (R, R) \subseteq N$.

Thererfore, $x\,(y, z) \in N$. This implies that $R\,(T, R) \subseteq N$.

Also $(T, R) \subseteq (R, R) \subseteq N$. So $(T, R) + R\,(T, R) \subseteq N$. That is, $R \subseteq N$. Hence R is associative. This contradicts the hypothesis.

So $(T, R) + R\,(T, R) = (0)$ which implies that $(T, R) = (0)$.

Theorem 3. Suppose R is simple, not associative and $(R, R) \subseteq N$.

Then $(R, R) \subseteq U$.

Proof. Taking $T = (R, R)$ in lemma 2, we see that $T \subseteq N$.

Therefore $(T, R) = (0)$. This implies that $((R, R), R) = (0)$. Hence

$$(R, R) \subseteq U.$$

3. CONSTRUCTION OF IDEALS

Lemma 4. Let A be a left ideal of R. Then

(i) $S \{ s \in A : sR \subseteq A \}$ is a two-sided ideal of R.

(ii) If $(R, R, R) \subseteq U$, then $(R, A, R) \subseteq S$.

Proof. (i) For any $s \in S$, $x \in R$, consider sx and xs. Let $y \in R$. Then

$$
\begin{aligned}
(s\,x)\,y &= (s, x, y) + s\,(x\,y) \\
&= -\ (x, y, s) - (y, s, x) + s\,(x\,y) \qquad \text{(using (2))} \\
&= -\ (x, y, s) - (y, x, s) + s\,(x\,y) \qquad \text{(using (1))}.
\end{aligned}
$$

A is a left ideal and $s \in S$ implies that

$$-\ (x, y, s) - (y, x, s) + s\,(x\,y) \in A.$$

Hence $(sx)\,y \in A$. Also $sR \subseteq A$ implies that $sx \in A$. Hence $sx \in S$. Now,

$$
\begin{aligned}
(x\,s)\,y &= (x, s, y) + x\,(s\,y) \\
&= (x, y, s) + x\,(s\,y) \qquad \text{(using (1))}.
\end{aligned}
$$

Again using the fact that A is a left ideal and $s \in S$ in the above equation, we get $(x\,s)y \in A$. Hence $xs \in S$. So S is a two sided ideal of R.

(ii) Let x, w, $z \in R$ and $a \in A$. Then

$$
\begin{aligned}
(w, a, x)\,z &= (w, x, a)\,z \qquad \text{(using (1))}. \\
&= z\,(w, x, a).
\end{aligned}
$$

Again, since A is a left ideal, $z(w, x, a) \in A$. Therefore $(w, a, x)\,z \in A$. Hence $(R, A, R) \subseteq S$.

Theorem 5. Suppose $(R, R) \subseteq L$ and $(R, R, R) \subseteq U$. If R has a maximal left ideal $A \neq (0)$, which contains no two-sided ideal of R other than (0), then R is associative.

Proof. By Lemma 4, S is a two-sided ideal of R contained in A. Therefore $S = (0)$ and hence $(R, A, R) = (0)$. Using (1), we get $(R, R, A) = (0)$. Using (2), we obtain $(A, R, R) = (0)$, Now as in the proof of Lemma 1, $A + AR$ is a two-sided ideal of R. Since $A \subset A + AR$, we must have $A + AR = R$, Hence as in the proof of Lemma 1, $(R, R, R) \subseteq (A, R, R) = (0)$. That is, R is associative.

Theorem 6. Suppose $(R, R) \subseteq N$ and let A be a left ideal of R.

(a) If A is maximal and nil, then A is a two-sided ideal of R.

(b) If A is maximal, then either A is a two-sided ideal of R or the ideal generated in R by A is contained in N.

Proof. The motivation for this theorem and its method of proof are same as Theorem 1 in [1]. In fact, if one exchanges (1, 1) for right alternative, A for L, R for A, and the identities employed, then the notaiton and write up of the proof in [1] is virtually identical to the one here. However, in proof of part (b), one may argue as follows:

Clearly $(R, R, A) \subseteq A$.

A straight forward verification shows that any ring satisfies

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z$$

which is known as the Teichmuller identity. Put $w = x, x = y$, $y = a, z = r$ in this identity where $a \in A$, $r \in R$. We get

$$(x, y, a)r = (xy, a, r) - (x, ya, r) + (x, y, ar) - x(y, a, r)$$
$$= (xy, r, a) - (x, r, ya) + (x, y, ra) - x(y, r, a).$$

So $(x, y, a)r \in A$. This implies that $(R, R, A) \subseteq A' = 0$, i.e. $A \subseteq N$.

4. LEFT PRIMITIVE RINGS

Definition. A left ideal A of R is called regular if there exists an element $g \in R$ such that $x - xg \in A$ for all $x \in R$. R is called **left pimitive** if it contains a regular maximal ideal which contains no two-sided ideal of R other than (0).

Theorem 7. Suppose $(R, R) \subseteq L$ and $(R, R, R) \subseteq U$. If R is a left primitive ring, then either R is associative or it is simple with a right identity element.

Proof. Let A be a regular maximal left ideal which contains no two-sided ideals of R other than (0). If $A \neq (0)$, then by Theorem 5, R is associative. Thus assume that (0) is a maximal regular left ideal of R, which contains no two-sided ideal of R other than (0). (0) is maximal implies that R has no proper ideal. Therefore R is simple. Since (0) is regular, there exists $g \in R$ such that $x - xg \in (0)$ for all $x \in R$. That is, $x = xg$ for all $x \in R$ and g is a right identity element. Hence R is simple with a right identity element.

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