Jñānābha, Vol. 23, 1993

(Dedicated to Professor S. A. Naimpally on his 62 th Birthday)

CERTAIN FIVE SERIES EQUATIONS INVOLVING GENERALISED LAGUERRE POLYNOMIALS

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(Received : September, 30, 1992 ; Revised : October 2, 1993)

ABSTRACT

Solution of certain five series equations, involving generalised Laguerre polynomials, has been obtained in this paper.

1. INTRODUCTION

Dual and triple series equations arise in many mixed boundary value problems of mathematical physics [8]. The above equations are the generalisation of dual, triple and quadruple series equations, considered earlier by [9], [2] and [3]. The object of this paper is to obtain the solution of five series equations involving generalised Laguerre polynomials. Recently various authors [1], [4], [5], [6] and [7] have considered other series equations involving different special functions.

2. THE EQUATIONS

We shall solve the following system of five series equations involving generalised Laguerre polynomials.

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+1)} L_n^{\nu}(x) = \begin{cases} f_1(x) \ , \ 0<x<a \\ f_3(x) \ , \ b<x<c \\ f_5(x) \ , \ d<x<\infty \end{cases} \qquad \ldots (2.1)$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)} L_n^{\sigma}(x) = \begin{cases} f_2(x) \ , \ a<x<b \\ f_4(x) \ , \ c<x<d \end{cases} \qquad \ldots (2.2)$$

where $L_n^{\alpha}(x)$ is the generalised Laguerre polynomial, A_n is unknown coefficient, $f_1(x), f_2(x), f_3(x), f_4(x)$ and $f_5(x)$ are known functions and the parameters $\alpha, \beta, \nu, \sigma$ all are > -1. It is shown here that the problem of solving five series equations can be reduced to that of solving simultaneous Fredholm integral equations of second kind.

3. SOLUTION OF FIVE SERIES EQUATIONS

Let us suppose

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+1)} L_n^{\nu}(x) = \begin{cases} \phi_1(x), & a<x<b \\ \phi_2(x), & c<x<d \end{cases} \quad \text{... (3.1)}$$

Using orthogonality relation, we get from (2.1) and (3.1)

$$A_n = \frac{\Gamma(n+1)\,\Gamma(\alpha+n+1)}{\Gamma(\nu+n+1)} \left[\int_0^a f_1(x) + \int_a^b \phi_1(x) + \int_b^c f_3(x) + \int_c^d \phi_2(x) \right.$$

$$\left. + \int_d^{\infty} f_5(x) \right] x^{\nu} e^{-x} L_n^{\nu}(x)\, dx \quad \text{... (3.2)}$$

Substituting this expression for A_n from (3.2) in (2.2) we have

$$\int_a^b e^{-r} S(r,x)\,\phi_1(r)\,dr + \int_c^d e^{-r} S(r,x)\,\phi_2(r)\,dr = \begin{cases} M(x), & a<x<b \quad \text{... (3.3)} \\ N(x), & c<x<d \quad \text{... (3.4)} \end{cases}$$

where

$$M(x) = x^{\sigma} f_2(x) - \int_0^a e^{-r} S(r,x) f_1(r) dr - \int_b^c e^{-r} S(r,x) f_3(r)\, dr$$

$$- \int_d^{\infty} e^{-r} S(r,x) f_5(r)\, dr \quad \text{... (3.5)}$$

$$N(x) = x^{\sigma} f_4(x) - \int_0^a e^{-r} S(r,x) f_1(r)\, dr - \int_b^c e^{-r} S(r,x) f_3(r)\, dr$$

$$- \int_d^{\infty} e^{-r} S(r,x) f_5(r)\, dr \quad \text{... (3.6)}$$

and

$$\Gamma(\lambda)\,\Gamma(1-\lambda)\,S(r,x) = \Gamma(\lambda)\,\Gamma(1-\lambda)\, r^{\sigma} x^{\nu}$$

$$\sum_{n=0}^{\infty} \times \frac{\Gamma(\beta+n+1)\,\Gamma(n+1)}{\Gamma(\alpha+n+1)\,\Gamma(\sigma+n+1)} L_n^{\sigma}(r)\, L_n^{\nu}(x) \quad \text{... (3.7)}$$

$$= a^* \int_0^t \eta(\xi)\,(r-\xi)^{\lambda-1}\,(x-\xi)^{\lambda+\nu-\sigma-1}\, d\xi \quad \text{... (3.8)}$$

$$= a^* \; S_r(r,x) \quad \text{... (3.9)}$$

It is further assumed that a^* is independent of n.

Starting with equation (3.3) and making some calculations, we derive

$$\int_a^x \frac{\eta(y)\,\Phi_1(y)}{(x-y)^{1+\nu-\lambda-\sigma}}\, dy = \frac{\Gamma(\lambda)\,\Gamma(1-\lambda)}{a^*} M(x) - \int_0^a \frac{\eta(y)}{(x-y)^{1+\nu-\lambda-\sigma}}\, dy$$

$$\times\int_a^b \frac{e^{-r}\,\phi_1(r)}{(x-y)^{1-\lambda}}\,dr - \int_0^x \frac{\eta(y)}{(x-y)^{1+\nu-\lambda-\sigma}}\,dy \int_c^d \frac{e^{-r}\,\phi_2(r)}{(r-y)^{1-\lambda}}\,dr \quad \text{... (3.10)}$$

where

$$\Phi_1(y) = \int_y^b \frac{e^{-r}\,\phi_1(r)}{(r-y)^{1-\lambda}}\,dr, \quad 0<\lambda<1 \quad \text{... (3.11)}$$

Equation (3.10) is an Abel integral equation and its solution is

$$\eta(y)\,\Phi_1(y) = F(y) - \frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi}\int_0^a \eta(\xi)\,d\xi$$

$$\times\frac{d}{dy}\int_a^y \frac{dx}{(y-x)^{\lambda+\sigma-\nu}(x-\xi)^{1+\nu-\lambda-\sigma}}$$

$$\times\int_a^b \frac{e^{-r}\,\phi_1(r)}{(r-\xi)^{1-\lambda}}\,dr - \frac{\sin(1+\nu-\lambda-\sigma)\,\pi}{\pi}\int_0^a \eta(\xi)\,d\xi\,\frac{d}{dy}\int_a^y$$

$$\times\frac{dx}{(y-x)^{\lambda+\sigma-\nu}(x-\xi)^{1+\nu-\lambda-\sigma}}\int_c^d \frac{e^{-r}\,\phi_2(r)}{(r-\xi)^{1-\lambda}}dr - \frac{\sin(1+\nu-\lambda-\sigma)\,\pi}{\pi}\,\frac{d}{dy}$$

$$\times\int_a^y \eta(\xi)\,d\xi\int_\xi^y \frac{dx}{(y-x)^{\lambda+\sigma-\nu}(x-\xi)^{1+\nu-\lambda-\sigma}}\int_c^d \frac{e^{-r}\,\phi_2(r)}{(r-\xi)^{1-\lambda}}\,dr. \quad \text{... (3.12)}$$

where

$$F(y) = \frac{\sin(1+\nu-\lambda-\sigma)\,\pi}{\pi}\,\frac{\Gamma(\lambda)\,\Gamma(1-\lambda)}{a^*}\,\frac{d}{dy}\int_a^y \frac{M(x)}{(y-x)^{\lambda+\sigma-\nu}}\,dx \quad \text{.. (3.13)}$$

With the help of equation (3.11), we obtain

$$\int_a^b \frac{e^{-r}\,\phi_1(r)}{(r-\xi)^{1-\lambda}}\,dr = \frac{\sin(1-\lambda)\,\pi}{\pi\,(a-\xi)^{-\lambda}}\int_a^b \frac{\Phi_1(y)}{(y-a)^{\lambda}\,(y-\xi)}\,dy \quad \text{... (3.14)}$$

Now let us consider

$$\Phi_2(y) = \int_y^d \frac{e^{-r}\,\phi_2(r)}{(r-y)^{1-\lambda}}\,dr\,. \qquad 0<\lambda<1 \quad \text{... (3.15)}$$

Therefore

$$\int_c^d \frac{e^{-r}\,\phi_2(r)}{(r-\xi)^{1-\lambda}}\,dr = \frac{\sin(1-\lambda)\,\pi}{\pi\,(c-\xi)^{-\lambda}}\int_c^d \frac{\Phi_2(y)}{(y-c)^{\lambda}\,(y-\xi)}\,dy \quad \text{... (3.16)}$$

Making an appeal to the results (3.14), (3.15) and (3.16), we derive from (3.12)-

$$\eta(y)\,\Phi_1(y) = F(y) - \int_a^b P(t,y)\,\Phi_1(t)\,dt - \int_c^d Q(t,y)\,\Phi_2(t)\,dt - \frac{\sin(1-\lambda)\pi}{\pi}$$

$$\times \frac{d}{dy}\int_a^y \frac{\eta(\xi)}{(c-\xi)^{-\lambda}}\,d\xi \int_c^d \frac{\Phi_2(t)}{(t-c)^{\lambda}(t-\xi)}\,dt \qquad \text{... (3.17)}$$

where

$$\frac{d}{dy}\int_a^y \frac{dx}{(y-x)^{\lambda+\sigma-\nu}(x-\xi)^{1+\nu-\lambda-\sigma}} = \frac{(a-\xi)^{\lambda+\sigma-\nu}}{(y-\xi)(y-a)^{\lambda+\sigma-\nu}} \qquad \text{... (3.18)}$$

$$\int_t^{\xi} \frac{dx}{(\xi-x)^{\lambda-\nu+\sigma}(x-t)^{1-\sigma-\lambda+\nu}} = \frac{\pi}{\sin(1+\nu-\lambda-\sigma)\pi} \qquad \text{... (3.19)}$$

$$P(t,y) = \frac{\sin(1+\nu-\lambda-\sigma)\pi\,.\,\sin(1-\lambda)\pi}{\pi^2(y-a)^{\lambda+\sigma-\nu}}\cdot\frac{1}{(t-a)^{\lambda}}\int_0^a \frac{(a-\xi)^{2\lambda+\sigma-\nu}\,\eta(\xi)}{(y-\xi)(t-\xi)}\,d\xi$$

$$\text{... (3.20)}$$

$$Q(t,y) = \frac{\sin(1+\nu-\lambda-\sigma)\pi\,.\,\sin(1-\lambda)\pi}{\pi^2(y-a)^{\lambda+\sigma-\nu}}\cdot\frac{1}{(t-c)^{\lambda}}$$

$$\int_0^a \frac{(a-\xi)^{\lambda+\sigma-\nu}(c-\xi)^{\lambda}\,\eta(\xi)}{(y-\xi)(t-\xi)}\,d\xi \qquad \text{... (3.21)}$$

Now starting with equation (3.4), we derive

$$\eta(y)\,\Phi_2(y) = G(y) - \frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi}\int_0^c \eta(\xi)\,d\xi\,\frac{d}{dy}\int_c^y$$

$$\times \frac{dx}{(x-\xi)^{1+\nu-\lambda-\sigma}(y-x)^{\lambda+\sigma-\nu}}\int_c^d \frac{e^{-r}\,\phi_2(r)}{(r-\xi)^{1-\lambda}}\,dr - \frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi}\,.$$

$$\times \int_0^a \eta(\xi)\,d\xi\,\frac{d}{dy}\int_c^y \frac{dx}{(x-\xi)^{1+\nu-\lambda-\sigma}(y-x)^{\lambda+\sigma-\nu}}\int_a^b \frac{e^{-r}\phi_1(r)}{(r-\xi)^{1-\lambda}}\,dr -$$

$$\frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi}\int_a^b \eta(\xi)\,d\xi\,\frac{d}{dy}\int_c^y \frac{dx}{(x-\xi)^{1+\nu-\lambda-\sigma}(y-x)^{\lambda+\sigma-\nu}}$$

$$\times \int_{\xi}^b \frac{e^{-r}\,\phi_1(r)}{(r-\xi)^{1-\lambda}}\,dr \qquad \text{... (3.22)}$$

where

$$G(y) = \frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi} \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a^*} \frac{d}{dy} \int_c^y \frac{N(x)}{(y-x)^{\lambda+\sigma-\nu}} dx \quad \text{.. (3.23)}$$

With the help of results (3.15), (3.16), (3.18) and (3.19), we also obtain

$$\eta(y)\Phi_2(y) = G(y) - \int_c^d R(t,y)\Phi_2(t)\,dt - \int_a^b S(t,y)\Phi_1(t)\,dt -$$

$$\frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi(y-c)^{\lambda+\sigma-\nu}} \int_a^b \frac{(c-\xi)^{\lambda+\sigma-\nu}\eta(\xi)}{(y-\xi)} d\xi \int_\xi^b \frac{e^{-r}\phi_1(r)}{(r-\xi)^{1-\lambda}} dr \quad \text{... (3.24)}$$

where

$$R(t,y) = \frac{\sin(1+\nu-\lambda-\sigma)\pi \,.\, \sin(1-\lambda)\pi}{\pi^2 (y-c)^{\lambda+\sigma-\nu}(t-c)^{\lambda}} \int_0^c \frac{(c-\xi)^{2\lambda+\sigma-\nu}\eta(\xi)}{(y-\xi)(t-\xi)} d\xi \quad \text{... (3.25)}$$

$$S(t,y) = \frac{\sin(1+\nu-\lambda-\sigma)\pi \,.\, \sin(1-\lambda)\pi}{\pi^2 (y-c)^{\lambda+\sigma-\nu}(t-a)^{\lambda}} \int_0^a \frac{(c-\xi)^{\lambda+\sigma-\nu}(a-\xi)^{\lambda}\eta(\xi)}{(y-\xi)(t-\xi)} d\xi \quad \text{... (3.26)}$$

The equations (3.17) and (3.24) are simultaneous Fredholm integral equations of second kind, which determine $\Phi_1(y)$ and $\Phi_2(y)$. $\phi_1(r)$ and $\phi_2(r)$ can then be found from equations (3.11) and (3.15) respectively and hence the coefficients A_n can be determined from the equation (3.2).

4. PARTICULAR CASES

From the above results, it is easy to derive the solutions of the corresponding dual, triple and quadruple series equation involving generalised Laguerre polynomials.

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