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(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

SOME FIVE SERIES EQUATIONS INVOLVING GENERALISED BATEMAN K- FUNCTIONS

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ABSTRACT

Solution of certain five series equations involving generalised Bateman K-functions, has been obtained in this paper.

1. INTRODUCTION

Dual and triple series equations arise in many mixed boundary value problems of mathematical physics [8]. The above equations are the generalisation of dual, triple and quadruple series equations considered earlier by [2] , [3] and [9]. Recently Dwivedi and Pandey [4] have obtained solution of the above set of equations but calculations are erroneous. Therefore the object of this paper is to find the solution of the same set of equations whose solutions appear to be correct. Various other authors [1], [5], [6] and [7] have considered other set of series equations involving different special functions.

2. THE EQUATIONS

We shall solve the following system of five series equations involving generalised Bateman K-functions :

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2\beta+\sigma+n+1)} K_{2\,(n+\alpha)}^{2\,(\alpha+\sigma)}(x) = \begin{cases} f_1(x) \ , & 0<x<a \\ f_3(x) \ , & b<x<c \\ f_5(x) \ , & d<x<\infty \end{cases} \qquad \text{... (2.1)}$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2\nu+\sigma+n+1)} K_{2\,(n+\beta)}^{2\,(\beta+\sigma)}(x) = \begin{cases} f_2(x) \ , & a<x<b \\ f_4(x) \ , & c<x<d \end{cases} \qquad \text{... (2.2)}$$

where $K_{2\,(n+\alpha)}^{2\,(\alpha+\sigma)}(x)$ is the generalised Bateman K-function, A_n is unknown coefficient, $f_1(x), f_2(x)$, $f_3(x)$, $f_4(x)$ and $f_5(x)$ are known functions and the parameters $\alpha, \beta, \nu, \sigma$ all are > -1. It is shown here that the problem of solving five series equations can be reduced to that of solving simultaneous Fredholm integral equations of second kind.

3. SOLUTION OF FIVE SERIES EQUATIONS

Let us suppose

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2\beta+\sigma+n+1)} K_{2(n+\alpha)}^{2(\alpha+\sigma)}(x) = \begin{cases} \phi_1(x), & a < x < b \\ \phi_2(x), & c < x < d \end{cases} \quad \ldots (3.1)$$

Using orthogonality relation, we get from (2.1) and (3.1)

$$A_n = \frac{\Gamma(2\beta+\sigma+n+1)\,\Gamma(2\alpha+\sigma+n+1)}{2^{2\alpha+2\sigma}\,\Gamma(n-\sigma)} \left[\int_0^a f_1(x) + \int_a^b \phi_1(x) + \int_b^c + f_3(x)\right.$$

$$\left. + \int_c^d \phi_2(x) + \int_d^\infty f_5(x)\right] x^{-2\alpha-2\sigma-1} K_{2(n+\alpha)}^{2(\alpha+\sigma)}(x)\,dx. \quad \ldots (3.2)$$

Substituting this expression for A_n from (3.2) in (2.2), we have

$$\int_a^b \phi_1(r)\,r^{-2\alpha-2\sigma-1} S(r,x)\,dr + \int_c^d \phi_2(r)\,r^{-2\alpha-2\sigma-1} S(r,x)\,dr$$

$$= \begin{cases} M(x), & a < x < b \quad \ldots (3.3) \\ N(x), & c < x < d \quad \ldots (3.4) \end{cases}$$

where,

$$M(x) = \frac{\{\Gamma(2\nu+\sigma+n+1)\}^2}{\Gamma(2\beta+\sigma+n+1)\,\Gamma(2\alpha+\sigma+n+1)} f_2(x) - \left[\int_0^a f_1(r) + \int_b^c f_3(r)\right.$$

$$\left. + \int_d^\infty f_5(r)\right] r^{-2\alpha-2\sigma-1} S(r,x)\,dr \quad \ldots (3.5)$$

$$N(x) = \frac{\{\Gamma(2\nu+\sigma+n+1)\}^2}{\Gamma(2\beta+\sigma+n+1)\,\Gamma(2\alpha+\sigma+n+1)} f_4(x)$$

$$- \left[\int_0^a f_1(r) + \int_b^c f_3(r) + \int_d^\infty f_5(r)\right] r^{-2\alpha-2\sigma-1} S(r,x)\,dr \quad \ldots (3.6)$$

and

$$S(r;x) = \sum_{n=0}^{\infty} \frac{\Gamma(2\nu+\sigma+n+1)}{2^{2\beta+2\sigma}\,\Gamma(n-\sigma)} K_{2(n+\beta)}^{2(\beta+\sigma)}(r)\; K_{2(n+\alpha)}^{2(\alpha+\sigma)}(x) \quad \ldots (3.7)$$

$$= \frac{e^{-x}\,2^{2\alpha-2\nu}}{\Gamma(2\alpha-2\nu)\,\Gamma(2\beta-2\nu)} \int_0^t E(\xi)\,(x-\xi)^{2\alpha-2\nu-1}\,(r-\xi)^{2\beta-2\nu-1}\,d\xi \quad \ldots (3.8)$$

$$= \frac{e^{-x}\,2^{2\alpha-2\nu}}{\Gamma(2\alpha-2\nu)\,\Gamma(2\beta-2\nu)} S_t(r,x) \quad \ldots (3.9)$$

Starting with equation (3.3) and making some calculations, we derive

$$\int_a^x \frac{E(\xi)\,\Phi_1(\xi)}{(x-\xi)^{1-2\alpha+2\nu}}\,d\xi = \frac{\Gamma(2\alpha-2\nu)\Gamma(2\beta-2\nu)}{2^{2\alpha-2\nu}} e^x M(x) - \int_0^a \frac{E(\xi)}{(x-\xi)^{1-2\alpha+2\nu}}\,d\xi$$

$$\times \int_a^b \frac{r^{-2\alpha-2\sigma-1}\phi_1(r)}{(r-\xi)^{1-2\beta+2\nu}}\,dr - \int_0^x \frac{E(\xi)}{(x-\xi)^{1-2\alpha+2\nu}}\,d\xi$$

$$\times \int_c^d \frac{r^{-2\alpha-2\sigma-1}\phi_2(r)}{(r-\xi)^{1-2\beta+2\nu}}\,dr \qquad \ldots (3.10)$$

where,

$$\Phi_1(\xi) = \int_\xi^b \frac{r^{-2\alpha-2\sigma-1}\phi_1(r)}{(r-\xi)^{1-2\beta+2\nu}}\,dr \qquad \ldots (3.11)$$

Equation (3.10) is an Abel integral equation and its solution is

$$E(\xi)\,\Phi_1(\xi) = F(\xi) - \frac{\sin(1-2\alpha+2\nu)\pi}{\pi}\int_0^a E(y)\,dy$$

$$\times \frac{d}{d\xi}\int_a^\xi \frac{dx}{(\xi-x)^{2\alpha-2\nu}(x-y)^{1-2\alpha+2\nu}}$$

$$\times \int_a^b \frac{r^{-2\alpha-2\sigma-1}\phi_1(r)}{(r-\xi)^{1-2\beta+2\nu}}\,dr - \frac{\sin(1-2\alpha+2\nu)\pi}{\pi}\int_0^a E(y)\,dy$$

$$\frac{d}{d\xi}\int_a^\xi \times \frac{dx}{(\xi-x)^{2\alpha-2\nu}(x-y)^{1-2\alpha+2\nu}}\int_c^d \frac{r^{-2\alpha-2\sigma-1}\phi_2(r)}{(r-\xi)^{1-2\beta+2\nu}}\,dr$$

$$-\frac{\sin(1-2\alpha+2\nu)\pi}{\pi}\frac{d}{d\xi}\int_a^\xi E(y)\,dy\int_y^\xi \frac{dx}{(\xi-x)^{2\alpha-2\nu}(x-y)^{1-2\alpha+2\nu}}$$

$$\times \int_c^d \frac{r^{-2\alpha-2\sigma-1}\phi_2(r)}{(r-\xi)^{1-2\beta+2\nu}}\,dr \qquad \ldots (3.12)$$

where,

$$F(\xi) = \frac{\sin(1-2\alpha+2\nu)\pi}{\pi}\frac{\Gamma(2\alpha-2\nu)\,\Gamma(2\beta-2\nu)}{2^{2\alpha-2\nu}}\frac{d}{d\xi}\int_a^\xi \frac{e^x M(x)}{(\xi-x)^{2\alpha-2\nu}}\,dx \qquad \ldots (3.13)$$

with the help of equation (3.11) we obtain

$$\int_a^b \frac{r^{-2\alpha-2\sigma-1}\phi_1(r)}{(r-y)^{1-2\beta+2\nu}}\,dr = \frac{\sin(1-2\beta+2\nu)\pi}{\pi(a-y)^{-2\beta+2\nu}}\int_a^b \frac{\Phi_1(\xi)}{(\xi-a)^{2\beta-2\nu}(\xi-y)}\,d\xi \qquad \ldots (3.14)$$

Now let us consider

$$\Phi_2(\xi) = \int_\xi^d \frac{r^{-2\alpha-2\sigma-1}\phi_2(r)}{(r-\xi)^{1-2\beta+2\nu}}\,dr. \qquad \ldots (3.15)$$

Therefore,

$$\int_c^d \frac{r^{-2\alpha-2\sigma-1}\,\phi_2(r)}{(r-y)^{1-2\beta+2\nu}}\,dr = \frac{\sin(1-2\beta+2\nu)\pi}{\pi\,(c-y)^{-2\beta+2\nu}} \int_c^d \frac{\Phi_2(\xi)}{(\xi-c)^{2\beta-2\nu}\,(\xi-y)}\,d\xi \qquad \text{... (3.16)}$$

Making an appeal to the above results (3.14), (3.15) and (3.16), we derive from (3.12)

$$E(\xi)\,\Phi_1(\xi) = F(\xi) - \int_a^b P(t,\xi)\,\Phi_1(t)\,dt - \int_c^d Q(t,\xi)\,\Phi_2(t)\,dt -$$

$$\frac{\sin(1-2\beta+2\nu)\pi}{\pi}\,\frac{d}{d\xi}\int_a^\xi \frac{E(y)}{(c-y)^{-2\beta+2\nu}}\,dy \int_c^d \frac{\Phi_2(t)}{(t-c)^{2\beta-2\nu}\,(t-y)}\,dt \qquad \text{... (3.17)}$$

where,

$$\frac{d}{d\xi}\int_a^\xi \frac{dx}{(\xi-x)^{2\alpha-2\nu}\,(x-y)^{1-2\alpha+2\nu}} = \frac{(a-y)^{2\alpha-2\nu}}{(\xi-y)\,(\xi-a)^{2\alpha-2\nu}}, \qquad \text{... (3.18)}$$

$$\int_t^\xi \frac{dx}{(\xi-x)^{2\alpha-2\nu}\,(x-t)^{1-2\alpha+2\nu}} = \frac{\pi}{\sin(1-2\alpha+2\nu)\pi}, \qquad \text{... (3.19)}$$

$$P(t,\xi) = \frac{\sin(1-2\alpha+2\nu)\pi\,.\,\sin(1-2\beta+2\nu)\pi}{\pi^2\,(\xi-a)^{2\alpha-2\nu}}\,.\,\frac{1}{(t-a)^{2\beta-2\nu}}$$

$$\int_0^a \frac{(a-y)^{2\alpha+2\beta-4\nu}\,E(y)}{(\xi-y)\,(t-y)}\,dy \qquad \text{... (3.20)}$$

and

$$Q(t,\xi) = \frac{\sin(1-2\alpha+2\nu)\pi\,.\,\sin(1-2\beta+2\nu)\pi}{\pi^2\,(\xi-a)^{2\alpha-2\nu}}\,.\,\frac{1}{(t-c)^{2\beta-2\nu}}$$

$$\times \int_0^a \frac{(a-y)^{2\alpha-2\nu}\,(c-y)^{2\beta-2\nu}\,E(y)}{(\xi-y)\,(t-y)}\,dy \qquad \text{... (3.21)}$$

Now starting with equation (3.4), we derive

$$E(\xi)\,\Phi_2(\xi) = G(\xi) - \frac{\sin(1-2\alpha+2\nu)\pi}{\pi}\int_0^c E(y)\,dy$$

$$\times \frac{d}{dy}\int_c^\xi \frac{dx}{(x-y)^{1-2\alpha+2\nu}\,(\xi-x)^{2\alpha-2\nu}}$$

$$\times \int_c^d \frac{r^{-2\alpha-2\sigma-1}\,\phi_2(r)}{(r-y)^{1-2\beta+2\nu}}\,dr - \frac{\sin(1-2\alpha+2\nu)\pi}{\pi}\int_0^a E(y)\,dy\,\frac{d}{d\xi}\int_c^\xi$$

$$\times \frac{dx}{(x-y)^{1-2\alpha+2\nu}\,(\xi-x)^{2\alpha-2\nu}} \int_a^b \frac{r^{-2\alpha-2\sigma-1}\,\phi_1(r)}{(r-y)^{1-2\beta+2\nu}}\,dr$$

$$- \frac{\sin (1 - 2\alpha + 2\nu) \pi}{\pi} \int_a^b E(y)\, dy \frac{d}{d\xi} \int_c^\xi \frac{dx}{(x-y)^{1-2\alpha+2\nu} (\xi - x)^{2\alpha-2\nu}}$$

$$\times \int_y^b \frac{r^{-2\alpha-2\sigma-1} \phi_1(r)}{(r-y)^{1-2\beta+2\nu}}\, dr \quad \ldots (3.22)$$

where

$$G(\xi) = \frac{\sin (1 - 2\alpha + 2\nu) \pi}{\pi} \frac{\Gamma(2\alpha - 2\nu)\, \Gamma(2\beta - 2\nu)}{2^{2\alpha - 2\nu}} \frac{d}{d\xi} \int_c^\xi \frac{e^x N(x)}{(\xi - x)^{2\alpha - 2\nu}}\, dx \quad \ldots (3.23)$$

With the help of results (3.15), (3.16), (3.18) and (3.19), we also obtain

$$E(\xi)\, \Phi_2(\xi) = G(\xi) - \int_c^d R(t, \xi)\, \Phi_2(t)\, dt - \int_a^b S(t, \xi)\, \Phi_1(t)\, dt$$

$$- \frac{\sin (1 - 2\alpha + 2\nu) \pi}{\pi (\xi - c)^{2\alpha - 2\nu}} \int_a^b \frac{(c - y)^{2\alpha - 2\nu}}{(\xi - y)} E(y)\, dy \int_y^b \frac{r^{-2\alpha-2\sigma-1} \phi_1(r)}{(r-y)^{1-2\beta+2\nu}}\, dr \quad \ldots (3.24)$$

where,

$$R(t, \xi) = \frac{\sin (1 - 2\alpha + 2\nu) \pi \, . \, \sin (1 - 2\beta + 2\nu) \pi}{\pi^2 (\xi - c)^{2\alpha - 2\nu} (t - c)^{2\beta - 2\nu}} \int_0^c \frac{(c - y)^{2\alpha + 2\beta - 4\nu} E(y)}{(\xi - y)(t - y)}\, dy \quad \ldots (3.25)$$

$$S(t, \xi) = \frac{\sin (1 - 2\alpha + 2\nu) \pi \, . \, \sin (1 - 2\beta + 2\nu) \pi}{\pi^2 (\xi - c)^{2\alpha - 2\nu} (t - a)^{2\beta - 2\nu}}$$

$$\int_0^a \frac{(c - y)^{2\alpha - 2\nu} (a - y)^{2\beta - 2\nu} E(y)}{(\xi - y)(t - y)}\, dy \quad \ldots (3.26)$$

The equations (3.17) and (3.24) are simultaneous Fredholm integral equations of second kind, which determine $\Phi_1(\xi)$ and $\Phi_2(\xi)$. $\phi_1(r)$ and $\phi_2(r)$ can then be found from equations (3.11) and (3.15) respectively and hence the coefficient A_n can be determined from the equation (3.2).

4. PARTICULAR CASES

From the above results, it is easy to derive the solutions of the corresponding dual, triple and quadruple series equations involving generalised Bateman K-function.

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