Jñānābha, Vol.23, 1993
(*Dedicated to Professor S.A.Naimpally on his 62th Birthday*)

A NEW "LOGISTIC" DIFFERENCE EQUATION

By

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(*Received : May 20, 1993; revised : June 8, 1993*)

ABSTRACT

The derivation of the usual logistic curve as a model of restricted growth such as the spread of rumour or disease depends on several unwarranted assumptions. In this note we consider the model in more detail an deduce and equation very similar to the usual one but seems to be better hehaved.

If a rumour or a disease is being spread in a population of size 100 by the mechanism "each person meets k other people each day - k being a constant, and spreads the rumour/disease at every opportunity, and no one ever forgets the rumour/recovers from the disease", and z_n is the number of people who know the rumour/have the disease at the beginning of the n-th day, then the difference equation

$$z_{n+1} = z_n + kz_n(100 - z_n)/100 \qquad \text{... (1)}$$

with an initial condition given by a value z_1, where $0 < z_1 < 100$, is commonly used to describe the spread, see [1] for example. (It is usually written in the form $u_{n+1} = au_n(1 - u_n)$, where $a = 1 + k$ and $u_n = kz_n/100(1 + k)$. The continuous analogue, where the spread is continuous in time, has the solution :

$$z(t) = 100/(1 + Ce^{-kt}) \qquad \text{... (2)}$$

(C is a constant determined by z(0)), the so-called "logistic curve".

First we will review how (1) is obtained. We state the argument in terms of rumour. At the beginning of the n-th stage, when z_n people know the rumour, the proportion of ignorant people in the population is $(100 - z_n)/100$, so each of the people meets k people during the subsequent time interval, of whom $k(100 - z_n)/100$ learn the rumour, and (1) results. This argument is obviously dubious : the second order effect - each person who learns the rumour during the n-th stage will pass it on to others during that stage (which increases the number who learn the rumour) - and the interaction of the z_n people with each other - the fact that each of the z_n, by spreading the rumour, decreases the probability for the next person (which decreases the number who learn the rumour) - are both ignored. Also, each of the z_n

people contacts k people out of 99 (ie, not himself) so the ratio should be $(99-(z_n-1))/99=(100-z_n)/99$, a small difference which does not address the main objections. This change results in this solution to the continuous analogue :

$$z(t)=100/(1+Ce^{-100kt/99}) . \qquad \dots (3)$$

The models (1) describes suggest that $z_n<100$ for all n, but a sequecne defined by (1) may overshoot 100 - this occurs if and only if k > 1, and certainly such a valure of k is not unrealistic. In this note we will show that the difference equation.

$$u_{n+1}=100-(100-u_n)(98/99)^{cu_n} \qquad \dots (4)$$

with $0<u_1<100$ and c = $-k/(100\ln 98/99)$ = $.985k$ provides a better description which does not overshoot 100. In fact, it is immediately clear that the sequence $\{u_n\}$ increases monotonically to 100, as one would expect, and there are no bifuractions, no chaos. Also, the solution to the continuous case is (2) with a slight change in k (see equation (8) below).

We phrase the argument in terms of rumour. Given u_n, to find u_{n+1} we suppose that the u_n knowledgeable people queue up, then the first person in the queue meets a random selection of k people in the population at large, then the second perosn meets k people, and so on, until each of the u_n people in the queue have met their random selection of k people. The rumour is passed on at every opportunity, but we will not consider "second order rumourmongering", so our value for u_{n+1} will necessarily be an underestimate of the true value.

Consider the first person in the queue. He selects one of 99 people of whom u_n-1 are knowledgeable; thus the person he meets has the probability of $(100-u_n)/99$ of being ignorant, so the rumour is passed on to $(100-u_n)/99$ people. For the next person met, the probability of him being ignorant is

$$(100-(u_n+(100-u_n)/99))/99$$

and so on. The first person passes the rumour on to

$$(100-u_n)/99+(100-(u_n+(100-u_n)/99))/99+$$

$$(1000-(u_n+(100-u_n)/99+(100-(u_n+(100-u_n)/99))/99+\dots$$

to k terms, and this is $(100-u_n)(1-(98/99)^k)$. ... (5)

The second person starts off with

$$u_n+(100-u_n)(1-(98/99)^k) \qquad \dots (6)$$

knowing the rumour, so the number of people who learn it from him is (5) with u_n replaced by (6). Continuing this for all u_n people in the queue, and adding up, we get

$$u_{n+1} = 100 - (100 - u_n)(98/99)^{ku_n}. \quad ..(7)$$

The differential equation arising form this is easily derived: if the rumour is being continuously spread and u(t) is the number of people who know the rumour at time t, then

$$\begin{aligned} du/dt &= lim\ (u(t+h) - (u(t)/h \\ &= lim\ (100 - u(t)t)(1 - (98/99)^{khu})/h \\ &= (100 - u)\ (-ln(98/99))ku \qquad \text{(l'Hospital)} \qquad ...\ (8) \end{aligned}$$

which is just the differential equation for the usual logistic curve, but with k replaced by $-100k\ ln(98/99), = 1.015k$. (It is surprising that (7) doesn't converge to (3) - evidently the second effect and/or the interaction effect do not vanish for short time intervals, but the discrepency is small: 1.0150 is near 100/99.)

ACKNOWLEDGEMENTS

I am indebted to Professor James T. Sandefur, Department of Mathematics, Georgetown University, Washington, DC, for his comments on this result.

REFERENCE

[1] E. Batschelet, *Introduction to Mathematics for Life Scientists,* Springer-Verlag, New York, Heidelberg, and Berlin, 1973.