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(Dedicated to the memory of Professor K. L. Singh)

ON A DISTRIBUTIONAL MEIJER-BESSEL TRANSFORM

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ABSTRACT

In this paper two new characterizations of a generalized Meijer Transform for distributions have been developed, using two transfrmations on the space of distribution viz the dilations U_n and the exponential shifts T^{-p}. The standard theorems on analyticity, uniqueness, invertibility and some standard operation transform formulas are proved, using the new characterization as the definition of the generalized Meijer Transform.

1. INTRODUCTION

A grneralization of Meijer transformation called Meijer-Bessel transform $F_{(s)}$ of the generalized function $f \in M_{\mu, a}$ (I), $I = (0, \infty)$ and defined as

$$F(s) = (2/\pi)^{\frac{1}{2}} \int_0^\infty (st)^{\nu + \frac{1}{2}} K_\mu (st) f(t) \, dt$$

$$\text{or, } F(s) = < f(t), (2/\pi)^{\frac{1}{2}} (st)^{\lambda + \frac{1}{2}} K_\mu (st) > \qquad (1.1)$$

have been studied by Agrawal [1] where $K_\mu(z)$ denotes the Bessel function of third kind and $K_\mu(z)$ has the integral representation

$$K_\mu(z) = \frac{\sqrt{\pi}}{\Gamma(\mu + \frac{1}{2})} \left(\frac{z}{2}\right)^\mu \int_{-1}^\infty (t^2-1)^{\mu - \frac{1}{2}} e^{-zt} \, dt$$

for $Re\ \mu > -\frac{1}{2}$, $Re\ z > 0$ [Erdèlyi, 5, p. 18]

Also $M'_{\mu,a}(I)$ is the dual of the testing function space $M_{\mu,a}(I)$ defined in this paper.

In this paper we give two new characterizations of the Meijer-Bessel transform for distribution with the help of the dilations U_n and the exponential shifts T^{-p} introduced earlier by Gesztelyi [6]. It is interesting to note here that Gesztelyi considered two transformations viz., dilations U_n and exponential shifts T^{-p} which are defined for ordinary function f, complex number p, and positive integer n by

$$U_n f(t) = n f(nt) \tag{1.3}$$

$$T^{-p} f(t) = e^{-pt} (f)(t) \tag{1.4}$$

Gesztelyi shows that whenever the sequence $\{U_n f\}$ converges (in the sense of Mikusinski csnvergence [9] the limit is necessarily a complex number. Also he proved that if f is a function which has Laplace transform at p, then the sequence of functions $\{U_n T^{-p} f(t)\}$ converges (in the Mikusinski sense) as $n \to \infty$ to the classical Laplace transform of f at p. He then defined the Laplace Transform of a Mikusinski operator x as the limit (whenever it exists in the sense of Mikusinski convergence) of the sequence $\{U_n T^{-p} x\}$ and shows that his definition generalizes the previous formulations of the Laplace transform Mikusinski operators of G-Doetsch [4] and V. A. Ditkin ([2] - [3]). Price [10] working on the same time introduced the Laplace transform of a distribution f using the sequence of the form $\{U_n T^{-p} f\}$ and showed that the new definition is equivalent to the Schwortz's extension of the transform to distributions He also introduced spaces B and B_0 and their duals B' and B'_0 has and shows that each distribution f in B'_0 has a unique extension $\check{f}$ in B'. He also shows that the sequence $\{U_j f\}$ converges $< f, 1 > \delta$, whenever f is in B'_0. Recently the present author ([7] - [8]) has given new characterizations for a generalized Laplace and Hermite transform for distributions.

In fact, B_0 is a subspace of $B(R^n)$ (or where R^n is understood, by B, space of all complex valued functions of an n-dimensional real variable $t = (t_1, t_2, \ldots, t_n)$ which possess contineous and bounded partial derivatives of all orders) consisting of those functions in B each of whose derivatives approaches to zero as $|t| \to \infty$, B'_0 (the dual of B_0) is a subspace of D' and a distribution f in B'_0 is completely determined by its value in D.

2. The Testing Function Spaces $M_{\mu, a}(I)$ and its dual $M'_{\mu, a}(I)$

The testing function space $W_{\alpha,\beta}(I)$, consists of infinitely differentiable complex valued function $\phi(t)$ defined on I $(0,\infty)$ such that

$$\rho^{\mu,\lambda}_{a,k}(\phi) = \sup_{0<t<\infty} \left| e^{at} \, t^{\mu-\lambda-\frac{1}{2}} \, S^{k}_{\mu\,\lambda\,t} \, \phi(t) \right|$$

is finite , where

$$S_{\mu,\lambda,t} \equiv t^{\lambda-\mu-\frac{1}{2}} \, Dt^{2\mu+1} \, Dt^{-\lambda-\mu-\frac{1}{2}}$$

and a is a real number and $Re\,\mu > 0$.

The topograpy over $M_{\mu, a}(I)$ is generated by the collection of seminorms $\left\{ \rho^{\mu,\lambda}_{a,k} \right\}_{k=0}^{\infty}$ The dual of $M_{\mu, a}(I)$ is denoted by $M'_{\mu,a}(I)$ which consists of all continuous linear functionals on $M_{\mu,a}(I)$.

3. Two New Characterizations of the Meijer-Bessl Transform:

In this section we give two new characterizations of the Meijer-Bessel Transform for one dimensional distributions.

We will say that a distribution f is Meijer-Bessel transformable if there exists an open interval (c,d) such that whenever $p = \frac{1}{t} \log \left[\left(\frac{2}{\pi} \right)^{\frac{1}{2}} (st)^{\lambda+\frac{1}{2}} K_\mu(st) \right]$ $(t \neq 0$ is a complex number) is a complex number with real part in (c,d), $T^{-p} f$ is a distribution in B'_0 where B'_0 is the dual of B_0, a subspace of D' as defined by Price [10].

If (c,d) is the largest such open interval then the set $\Omega = \{p: Re\ p \in (c,d)\}$ is called the domain of the definition of the Meijer-Bessel traasform of f.

If f is a Meijer-Bessl transformable distribution where the transform has domain of definition Ω, then for $p \in \Omega$, we define the Meijer-Bessel transform $MB[f](p)$ of f at p by

$$MB[f](p) = \frac{1}{\phi(0)} \lim_{j \to \infty} < U_j\, T^{-p} f, \phi > \tag{3.1}$$

where ϕ is a test function in D with $\phi(0) \neq 0$.

We have another characterization also as

$$MB[f](p) = < T^{-p} f, 1 > \tag{3.2}$$

where $p = -\frac{1}{t} \log [(2/\pi)^{\frac{1}{2}} (st)^{\lambda+\frac{1}{2}} K_\mu(st)]$.

From (3.2) we see that $MB(f)(p)$ is a complex valued function of the complex variable p with domain Ω. In fact, the mapping MB is linear. For if f and g are distributions that are transformable at p and α, β are oomplex numbers then $(\alpha f + \beta g)$ is Meijer-Bessel transformable at p and

$$\begin{aligned} MB[\alpha f + \beta g](p) &= < T^{-p} (\alpha f + \beta g), 1 > \\ &= \{\alpha < T^{-p} f, 1 > + \beta < T^{-p} g, 1 >\} \\ &= \alpha\, MB[f](p) + \beta\, MB[g](p). \end{aligned}$$

Theorem 3.1: (Analyticity theorem):

If f is a distribution that is Meijer-Bessel transformable in Ω, then $MB[f](p)$ is analytic function of p in Ω and

$$\frac{d}{dp} MB[f](p) = MB[-t f(t)](p) \tag{3 3}$$

Proof: Let $\Omega = \{p: Re\ p \in (c,d)\}$ and

$$p = -\frac{1}{t} \log [(2/\pi)^{\frac{1}{2}} (st)^{\lambda+\frac{1}{2}} K_\mu(st)];\ t \neq 0,$$

Let us choose p_0 in Ω and ϵ in (0,1) such that
$\varepsilon < min \{Re\ p_0 - c, d - Re\ p_0\}$

If $\lambda(t) = e^{\varepsilon t} + e^{-\varepsilon t}$ then $\frac{1}{\lambda}$ is in $S \subset B_0$

and $(\lambda T^{-p_0} f)$ is in B'_0. Also as long as $|p - p_0| < \epsilon$, we have
$MB[f](p) - MB[f](p_0)$

$$= < \frac{e^{-pt} - e^{-p_0 t}}{p - p_0} f(t), 1(t) >$$

$$= < \lambda(t) e^{-p_0 t} f(t), \frac{1}{\lambda(t)} \left[\frac{e^{-(p-p_0)t}}{p - p_0} \right] >$$

$$= < \lambda(t) e^{-p_0 t} f(t), - \frac{1}{\lambda(t)} + \frac{(p-p_0)t^2}{\lambda(t)} \sum_{j=2}^{\infty} \frac{[-(p-p_0)t]^{j-2}}{j!} >$$

But each derivative of

$\frac{t^2}{\lambda(t)} \sum_{j=2}^{\infty} \frac{[-(p\ p_0)t]^{j-2}}{j!}$ is bounded in absolute value by the corresponding derivative of $\frac{t^2}{\lambda(t)}$ *exp.* $|(p - p_0)t|$ and is, therefore, in S. Thus as $p \rightarrow p_0$. $\frac{1}{\lambda(t)} \left[\frac{e^{-(p-p_0)t} - 1}{p\ p_0} \right]$ converges in B_0 to $- \frac{t}{\lambda(t)}$ and we have

$$\frac{d}{dp_0} MB[f](p_0) = \lim_{p \rightarrow p_0} \frac{MB[f](p) - MB[f](p_0)}{p\ p_0}$$

$$= < \lambda(t) T^{-p_0} f(t), - \frac{t}{\lambda(t)} >$$

$$= < T^{-p_0} [-t f(t), 1(t) >$$

$$= MB[\ t f(t)] (p_0)$$

4. Treatment of the Convolution of Two Distributions:

Much of the usefulness of Meijer-Bessel transform is a result of the way it treats the convolution of two distributions. We give here this important property of the transformation by the following theorem.

Theorem 4.1 : If f and g are Meijer-Bessel transformable distributions such that the domain of their tespective transforms have intersection Ω, then $f*g$ is Meijer-Bessel transformable in Ω and for every $p\epsilon\Omega$, we have.

$$MB[f*g](p) = MB[f](p)\,MB[g](p).$$

Proof: For $p\epsilon\Omega$, $T^{-p}f$ and $T^{-p}g$ are both in B'_0. So by Lemma 2.3 (p. 20, Price [10] whica states that if f and g are in $B'_0(R^n)$, then their convolution can be defined also in $B'_0(R_n)$ we have

$$T^{-p}f * T^{-p}g = T^{-p}(f+g) \text{ is in } B'_0.$$

Therefore, $(f_* g)$ is Meijer-Bessel transformable at p and from (3.2) and the definition of convolution, we get

$$\begin{aligned} MB[f*g](p) &= \langle T^{-p}(f*g), 1 \rangle \\ &= \langle T^{-p}f * T^{-p}g, 1 \rangle \\ &= \langle T^{-p}f(t) \otimes T^{-p}g(\tau), 1(t+\tau) \rangle \\ &= \langle T^{-}_{p}f(t) \otimes T^{-p}g(\tau), 1(t)\, 1(\tau) \rangle \\ &= \langle T^{-p}f, 1 \rangle \langle T^{-p}g, 1 \rangle \\ &= MB[f](p)\, MB[g](p). \end{aligned}$$

5. Inversion and Uniqueness Theorems for the Meijer-Bessel Transformation:

No theory of the Meijer-Bessel transform woud be useful without the inversion and uniqueness theorems. We give Thorem 5.1 which includes both inversion and uniqueness theorems as its corollary.

In what follows we will have as independent variable at various times the real variable t and real and imaginary parts of complex variable p.

$$p = -\frac{1}{t}\log\left[(2/\pi)^{\frac{1}{2}}(st)^{\lambda+\frac{1}{2}}K_{\mu}(st)\right];\ 0 < s < \infty,$$

$t \neq 0$. For this reason we will some time indicate particular independent variable for a space or an operation by a subscript ξ

e. g. $< f(\xi), e^{-i\omega\xi} >_{\xi}$, where $f(\xi)$ is in B'_0 and ω is a parameter.

Theorem 5.1: If f is a distribution in B'_0 then

$$f(t) = \frac{1}{2\pi}\lim_{r\to\infty}\int_{-r}^{r} e^{i\omega t} < f(\xi), e^{-i\omega\xi} >_{\xi} d\omega \qquad (5.1)$$

where the limit is taken in D'_t.

Proof: The proof is the same as that of Laplace transform as given in [10].

There are three corollaries of this theorem.

Corollary 5.1 : (a) If σ is a real number such that $e^{-\sigma t} f(t)$ is in $B'_{0\xi}$, then as distributions,

$$f(t) = \lim_{r\to\infty}\frac{1}{2\pi i}\int_{\sigma-ir}^{\sigma+ir} e^{pt} < e^{-p\xi} f(\xi), 1(\xi) >_{\xi} dp.$$

Proof: If $e^{-\sigma t} f(t) \in B'_{0\xi}$, then as long as $Re.\ p=\sigma$, $e^{-pt} f(t) \in B'_{0\xi}$, and

$$e^{-\sigma t} f(t) = \frac{1}{2\pi}\lim_{r\to\infty}\int_{-r}^{r} e^{i\omega t} < e^{-\sigma\xi} f(\xi), e^{-i\omega\xi} >_{\xi} d\omega$$

Thus we have,

$$f(t) = \frac{1}{2\pi}\lim_{r\to\infty}\int_{-r}^{r} e^{\sigma t} e^{i\omega t} < e^{-\sigma\xi} f(\xi), e^{-i\omega\xi} >_{\xi} d\omega$$

$$= \frac{1}{2\pi i}\lim_{r\to\infty}\int_{\sigma-ir}^{\sigma+ir} e^{pt} < e^{-p\xi} f(\xi), 1(\xi) >_{\xi} dp.$$

Corollary 5.2: (Inversion Theorem)

If f is a Meijer-Bessel transformable in

$$\Omega = \{ p: c < Re\, p < d \},\ p = -\frac{1}{t} \log [(2/\pi)^{\frac{1}{2}} . (st)^{\lambda+\frac{1}{2}} K_\mu (st)]$$

is a complex number, $t \neq 0$

then as long as $c < \sigma < d$,

$$f(t) = \lim_{r \to \infty} \frac{1}{2\pi i} \int_{\sigma - ir}^{\sigma + ir} e^{pt}\, MB[f] (p)\, dp.$$

where the limit is taken in $D_t^{'}$.

Corollary 5.3 : (Uniqueness Theorem)

If f and g are Meijer-Bessel transformable distributions such that $MB[f] (p) = MB(g) (p)$ on some vertical line in the common domain of the transform of f and g then $f = g$ as distributions.

6. Some Standard Operation Transform Formulae for the Distribution Meijer-Bessel Transform :

We will introduce some standard operation transform formulae using the characterizations of the transform given in (3.1) as below.

Let f be a Meijer-Bessel transformable distribution whose transform has domain of definition $\Omega = \{ p: c < Re\, p < d\}$,

where

$$p = -\frac{1}{t} \log [(2/\pi)^{\frac{1}{2}} (st)^{\lambda+\frac{1}{2}} K_\mu(st) f(t)\, dt \quad (s \neq 0)$$

is a complex number. Then $f^{(1)}$ is also Meijer-Bessel transformable in Ω. To compute the transform of $f^{(1)}$, let ϕ be a function in D such that ϕ (0) = 1, and let $j = 0$ be an integer. Then if $p \in \Omega$, we have

$$< U(T^{-p} f^{(1)}, \phi > = < f^{(1)} (t), e^{-pt} \phi (\frac{t}{j}) >$$

$$= < f(t), pe^{-pt} \phi (\frac{t}{j}) - \frac{1}{j} e^{-pt} \phi^{(1)} (\frac{t}{j}) >$$

$$= p < U_j T^{-p} f, \phi > - \frac{1}{j} < U_j T^{-p} f, \phi^{(1)} >$$

As $j \to \infty$, the term $\frac{1}{j} < U_j T^{-p} f, \phi^{(1)} >$ converges to zero and so from (3.1) we have

$$MB[f^{(1)}](p) = \lim_{j \to \infty} p < U_j T^{-p} f, \phi > = p\, MB[f](p) \qquad (6.1)$$

By induction we can prove that for every positive n,

$$MB[f^{(n)}(p) = p\, MB[f^{(n-1)}](p) = p^n MB[f](p) \qquad (6.2)$$

Another operation transform formula can be obtained from (3.3) viz.

$$MB[-t f(t)](p) = \frac{d}{dp} MB[f](p).$$

This formula can be extended by the method of induction for every positive integer n, to get

$$MB[t^n f(t)](p) = (-1)^n \frac{d^n}{dp^n} MB[f](p) \qquad (6.3)$$

If f is Meijer–Bessel transformable in Ω, then $f(t-\eta)$ is Meijer–Bessel transformable in Ω for every real number η and we have

$$< U_j T^{-p} f(t-\eta), \phi(t) > = < f(t-\eta), e^{-pt} \phi(\frac{t}{j}) >$$

$$= < f(t), e^{-p(t+\eta)} \phi(\frac{t+\eta}{j}) >$$

$$= e^{-p\eta} \lessdot U_j T^{-p} f(t), \phi(t+\eta) \gtrdot$$

Non, $\phi(t+\eta) \in D$ and as 1 on g as $\phi(\eta) \neq 0$,

$$\lim_{j \to \infty} e^{-p\eta} < U_j T^{-p} f(t), \phi(t+\eta) > =$$

$$\frac{1}{\phi(\eta)} e^{-p\eta} \lessdot T^{-p} f, 1 \gtrdot < \delta(t), \phi(t+\eta) >,$$

So we have $MB[f(t-\eta)](p) = e^{-p\eta} MB[f](p)$.

If q is a fixed complex number and f is Meijer–Bessel transformable in Ω then $e^{-qt} f(t)$ is Meijer–Bessel transformable in $\Omega' = \{p: c\text{-}Re\, q \lessdot Re\, p < d - Re\, q\}$, and we have

$$< U_t T^{-p} [e^{-qt} f(t)], \phi(t) >$$
$$= < U_t T^{-(p+q)} f, \phi >$$

Thus, whenever $p \in \Omega'$,

$MB [e^{-qt} f(t)] (p) = MB [f] (p).$

If k is a fixed positive integer and f is Meijer-Bessel transformable in Ω, then $(U_k f)$ is Meijer-Bessel transformable in $\Omega'' = \{ p: kc < Re\, p < kd \}$. For $p \in \Omega''$ we have

$$< U_t T^{-p} \{U_k f\}, \phi > = < U_k f, e^{-pt} \phi (\frac{t}{j}) >$$
$$= < f(t), e^{-pt/k} \phi (t/jk) >$$
$$= < U_t T^{-p/k} f, \phi (t/k) >$$

As $j \to \infty$, this converges to

$$< T^{-p/k} f, 1 > < \delta, \phi >.$$

So, we get the formula.

$MB [U_k f] (p) = MB [f] (p/k).$

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