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(Dedicated to the memory of Professor K. L. Singh)

INTEGRATION OF CERTAIN H-FUNCTIONS OF SEVERAL VARIABLES WITH RESPECT TO THEIR PARAMETERS

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ABSTRACT

The object of the present paper is to obtain some interesting results by integrating the multivariable H-function with respect to its parameters. Such integrals are useful in the study of certain boundary value problems.

1. INTRODUCTION

The H-function of several complex variables represented by means of the multiple contour integral was defined by H. M. Srivastava and R. Panda in a series of papers ([2] to [6]). We represent it as follows (see also Srivastava et al. [7, p. 251])

$$(1.1)\quad H[z_1,\ldots,z_r] = H^{0,\,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix}(a_j:\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,p}:(c'_j,\gamma'_j)_{1,p_1};\ldots;(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ (b_j:\beta'_j,\ldots,\beta^{(r)}_j)_{1,q}:(d'_j,\delta'_j)_{1,q_1};\ldots;(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r}\end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi w)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\cdots\phi_r(\xi_r)\,\psi(\xi_1,\ldots,\xi_r)\,z_1^{\xi_1}\cdots z_r^{\xi_r}\,d\xi_1\ldots d\xi_r.$$

where $w = \sqrt{-1}$

$$(1.2)\quad \phi_i(\xi_i) = \frac{\prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i) \prod_{j=1}^{n_i} \Gamma(1-c_j^{(i)} + \gamma_j^{(i)} \xi_i)}{\prod_{j=m_{i+1}}^{q_i} \Gamma(1-d_j^{(i)} + \delta_j^{(i)} \xi_1) \prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} \xi_1)}$$

$$(\forall\ i = 1,\ldots,r)$$

$$(1.3)\quad \psi(\xi_1,\ldots,\xi_r) = \frac{\prod_{i=1}^{n} \Gamma(1-a_j + \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i)}{\prod_{j=n+1}^{p} \Gamma(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i) \prod_{j=1}^{q} \Gamma(1-b_j + \sum_{i=1}^{r} \beta_j^{(i)} \xi_i)}$$

Conditions of convergence etc. can be seen in the papers of Srivastava and Panda referred to above. These conditions will be assumed to hold good in the multivariable H-function occurring in this paper.

As the multivarlable H function is the most general special function and most of the functions occurring in pure and applied mathematics are partcular cases of it, a number of results, known or new, can be derived easily from our results.

2. EVALUATION OF THE INTEGRALS

$$(2.1)\quad I = \frac{1}{2\pi_i} \int_{-w\infty}^{+w\infty} \Gamma(a+x)\,\Gamma(b-x)\,\Gamma(c-x)\; e^{\pm w\pi x}$$

$$H^{0,\ n:\ m_1,n_1;\ldots;\ m_r,n_r}_{p,q+1:\ p_1,q_1;\ \ldots;\ p_r,q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (a_j:\alpha'_j,\ldots;\alpha_j^{(r)})_{1,p} \\ (b_j:\beta'_j,\ldots;\beta_j^{(r)})_{1,q} \end{matrix}\right.$$

$$\left.\begin{matrix} & : (c'_j,\gamma'_j)_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (1-d+x:h_1,\ldots,h_r) & : (d'_j,\delta'_j)_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{matrix}\right] dx$$

$$= \Gamma(a+b)\,\Gamma(a+c)\,exp\,(\pm a\pi)\, H^{0,\ n+1\ \ :\ m_1,n_1;\ldots;m_r\,n_r}_{p+1,\ q+2:\ p_1,q_1;\ldots;\ p_r,q_r}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \left| \begin{matrix} (1+a+b+c-d:h_1,\ldots,h_r),\ (a_j:\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,p}: \\ (1+b-d:h_1,\ldots,h_r),\ (1+c-d:h_1,\ldots,h_r),\ (b_j,\beta'_j,\ldots,\beta^{(r)}_j)_{1,q}: \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (c'_j,\gamma'_j)_{1,p_1};\ldots;(C^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ (d'_j,\delta'_j)_{1,q_1};\ldots;(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r} \end{matrix} \right]$$

provided that (1) $h_i > 0$ $(i=1,2,\ldots,r)$

$$(2)\ Re\left[d-a-b-c+h_i\, d^{(i)}_j / \delta^{(i)}_j\right] > 0,$$

$$i = 1,2,\ldots,r;\quad j = 1,2,\ldots,m_i$$

Proof: In the integrand of (2.1) we replace the H-function of several variables by (1.1); change the order of integration which is justified under the conditions stated above, we get

$$= \frac{1}{(2\pi w)^r} \int_{L_1}\ldots\int_{L_r} \phi_1(\xi_1)\ldots\phi_r(\xi_r)\,\psi(\xi_1,\ldots,\xi_r)\ z_1^{\xi_1}\ldots z_r^{\xi_r}\ d\xi_1\ldots d\xi_r$$

$$\frac{1}{2\pi w}\int_{-w\infty}^{+w\infty} \frac{\Gamma(a+x)\,\Gamma(c-x)\,\Gamma(b-x)}{(d\ x+h_1\xi_1\ldots+h_r\xi_r)}\ e^{\mp w\pi x}\ dx.$$

Now we evaluate the inner integral using a known integral formula [8, p. 289], noting that it is a hypergeometric function with unit argument. On expressing the resulting expression with the help of (1.1), we obtain (2.1).

Proceeding similarly we easily evaluate the following integrals.

$$(2.2)\quad \frac{1}{2\pi w}\int_{-w\infty}^{+w\infty} \Gamma(a+x)\,\Gamma(b-x)\,exp(\mp w\pi x)$$

$$H^{0,\,n+1\,:\,m_1,n_1;\ldots;\,m_r,n_r}_{p+1,q+1:\,p+1,q1;\ldots;\,p_r,\,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \left| \begin{matrix} (1-c+x:h_1,\ldots,h_r), \\ (1-d+x:k_1,\ldots,k_r), \end{matrix} \right. \right.$$

$$\left.\begin{matrix} (a_j:\alpha'_j,\dots,\alpha^{(r)}_j)_{1,p} :(c'_j,\gamma'_j)_{1,p_1};\dots;(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ (b_j:\beta'_j,\dots,\beta^{(r)}_j)_{1,q} :(d'_j,\delta'_j)_{1,q_1};\dots;(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r} \end{matrix}\right] dx$$

$$= \Gamma(a+b)\, exp\,(\pm wa\pi)\, H^{0,\,n+2\,:\,m_1,\,n_1;\dots;m_r,n_r}_{p+2,\,q+2:\,p_1,\,q_1;\dots;\,p_r,q_r}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\right| \begin{matrix} (1-a-c:\,h_1,\dots,h_r),\,(1+a+b+c-d:\,k_1-h_1,\dots,k_r-h_r), \\ (1+b-d:\,k_1,\dots,k_r),\,(1+c-d:\,k_1-h_1;\dots;k_r-h_r), \end{matrix}$$

$$\left.\begin{matrix} (a_j:\alpha'_j,\dots,\alpha^{(r)}_j)_{1,p} : c'_j,\gamma'_j)_{1\,p_1},\dots,(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ b_j:\,j',\dots,j^{(r)})_{1,q} : d'_j,\delta'_j)_{1,q_1},\dots,(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r} \end{matrix}\right]$$

provided that

1. $k_i > h_i > 0$ for $i = 1,2,\dots,r$

2. $Re\left[\, d-a-b-c + (k_i-h_i)\, d^{(i)}_j / \delta^{(i)}_j \right] > 0$

   for $i = 1,2,\dots,r;\quad j = 1,2,\dots,m_i$

$$(2.3)\ \frac{1}{2\pi w}\int_{-w\infty}^{+w\infty} \frac{\Gamma(a+x)\,\Gamma(b-x)}{\Gamma(c-x)}\, exp\,(\mp w\pi x)\, H^{0,n+1:m_1,n_1,\dots,m_r,n_r}_{p+1,q:\,p_1,q_1,\dots,p_r,q_r}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\right| \begin{matrix} (1-d+x:\,h_1,\dots,h_r),\,(a_j:\,\alpha'_j,\dots,\alpha^{(r)}_j)_{1,p} \\ (b_j:\,\beta'_j,\dots,\beta^{(r)}_j)_{1,q} \end{matrix}$$

$$\left.\begin{matrix} :(c'_j,\gamma'_j)_{1,p_1},\dots,(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ :(d'_j,\delta'_j)_{1,q_1},\dots,(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r} \end{matrix}\right] dx$$

$$= \frac{\Gamma(a+b)\,\Gamma(c-a-b-d)}{\Gamma(c-b)}\, exp\,(\pm w\pi a)\, H^{0,\,n+1:\,m_1\,n_1,\dots,m_r,n_r}_{p+2,q+1:\,p_1,q_1,\dots,\,p_r,\,q_r}$$

$$\left[\begin{matrix} (-z_1)^{h_1} \\ \vdots \\ (-z_r)^{h_r} \end{matrix}\right| \begin{matrix} (1-a-d:h_1,\dots,h_r),\,(a_j,\,\alpha'_j,\dots,\alpha^{(r)}_j)_{1,p}, \\ (b_j,\,\beta'_j,\dots,\beta^{(r)}_j)_{1,q}, \end{matrix}$$

$$\begin{matrix} (c\text{-}d:h_1,\dots,h_r) & :(c'_j,\gamma'_j)_{1,p_1};\dots,(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ (c\text{-}a\text{-}b\text{-}d:h_1,\dots,h_r) & :(d'_j,\delta'_j)_{1,q_1},\dots;(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r} \end{matrix} \Bigg]$$

provided that

1. $h_i > 0$, for $i = 1,2,\dots,r$

2. $Re\left[c-a-b-d+\max h_i\left(\frac{c^{(i)}_j - 1}{\gamma^{(i)}_j}\right)\right] > 0$

for $i = 1,2,\dots,r;\quad j = 1,2,\dots,n_i$.

3. PARTICULAR CASES

(i) If we take all greek letters equal to unity in (1.1), the multivariate H-function would reduce to the relatively more familiar G-function of two variables. Thus our results (2.1), (2.2) and (2.3) will yield similar integrals involving the double G-function.

(ii) If we set $r = 2$, $m_2 = 1$, $q_2 = q_2 + 1$ in (2.1) we get an integral relation for the H-function of two variables in the form:

$$\frac{1}{2\pi w}\int_{-w\infty}^{+w\infty}\Gamma(a+x)\,\Gamma(b-x)\,\Gamma(c-x)\exp(\mp\pi x)\,H^{0,n:m_1,n_1,m_2,n_2}_{p,q:p_1,q_1,p_2,q_2}$$

$$\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| \begin{matrix} (a_j:\alpha'_j,\alpha''_j)_{1,p} & ; \\ (b_j:\beta'_j,\beta''_j)_{1,q};\ (1+d+x:h_1,h_2); & \end{matrix}\right.$$

$$\left.\begin{matrix} (c'_j,\gamma'_j)_{1,p_1};\ c''_j,\gamma''_j)_{1,p_2} \\ (d'_j,\delta'_j)_{1,q_1};\ (d''_j,\delta''_j)_{1,q_2} \end{matrix}\right]dx$$

$$= \Gamma(a+b)\,\Gamma(a+c)\exp(\pm a\pi)\,H^{0,n+1:m_1,n_1;1,n_2}_{p+1,q+2:p_1,q_1;p_2,q_2+1}$$

$$\left[\begin{matrix} z_1 \\ z_2 \end{matrix} \middle| \begin{matrix} (1+a+b+c-d : h_1, h_2) , \\ (1+b-d:h_1, h_2), \quad (1+c-d : h_1, h_2), \end{matrix}\right.$$

$$\left.\begin{matrix} (a_j:\alpha'_j, \alpha''_j)_{1,p} :(c'_j, \gamma'_j)_{1,p_1} ; (c''_j, \gamma''_j)_{1,p_2} \\ (b_j:\beta'_j, \beta''_j)_{1,q} :(d'_j, \delta'_j)_{1,q_1} ; (d''_j, \delta''_j)_{1,q_2} \end{matrix}\right] \quad \ldots (3.1)$$

with the conditions

1. $h_1 > 0, h_2 > 0$

2. $Re \left[d-a-b-c+h_1 \sum_{j=1}^{m_1} \frac{d'_j}{\delta'_j} + h_2 \sum_{j=1}^{m_2} \frac{d''_j}{\delta''_j} \right] > 0$

(iii) In (3.1), if we let $p=q=n=0$, $n_2=p_2=0$, $m=1=q_2$, $d''_1=0$, $\delta''_1=0$ and take limit $z_2 \to 0$; and arranging the parameters we get an integral relation proved earliear by Nair and Nambudripad [1].

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