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(Dedicated to the memory of Professor K. L. Singh)

A PARTICULAR ASYMPTOTIC BEHAVIOUR OF THE ZEROS OF σ- ORTHOGONAL POLYNOMIALS

By

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ABSTRACT

We study the asymptotic behaviour of the unique solution $(t_1^{(m)},\ldots,t_m^{(m)})$ of the system

$$\int_{-1}^{1} p(x)x^k \prod_{i=1}^{m} (x-t_i)^{2s_i+t}\, dx \qquad (k=0,1,\ldots,m-1),$$

satisfying the condition $-1 < t_1^{(m)} < t_2^{(m)} \leq \ldots \leq t_m^{(m)} \leq 1$ for every fixed $m \in N$, when only one of the integers s_i tends to infinity.

1. INTRODUCTION. In this paper we are concerned with the asymptotic behaviour of the unique solution $(t_1^{(m)},\ldots,t_2^{(m)},\ldots,t_m^{(m)})$ of the system

$$(1.1) \quad \int_{-1}^{1} p(x)\, x^k \prod_{i=1}^{m} (x-t_i)^{2s_i+1}\, dx=0, \quad (k=0,1,\ldots,m-1),$$

satisfying the condition

$$(1.2) \quad -1 < t_1^{(m)} < t_2^{(m)} \leq \ldots \leq t_m^{(m)} \leq 1.$$

The system (1.1) arises when we consider a generalization of Turan quadrature formula i. e. a quadrature formula with nodes having arbitrary multiplicity:

$$(1.3) \quad \int_{-1}^{1} p(x) f(x)\, dx = \sum_{i=1}^{m} \sum_{k=0}^{2s_i} A_{ik}\, f^{(k)}(x_i) + R[f]$$

where $p(x) \in L[-1,1]$, $p(x) > 0$ i. e. in $[-1,1]$, m is an arbitrary positive integer, $s_1, s_2, \dots, s_m$ are the first m terms of a sequence σ of nonnegative integers.

If we assume $-1 < x_1 < x_2 < \dots < x_m < 1$ the quadrature formula (1.3) attains tne maximum degree of accuracy $n = 2\left(\sum_{i=1}^{m} s_i + m\right)$ (It means that $R[f] = 0$ when f is an arbitrary polynomial of degree $\leqslant n-1$) if and only if $(x_1, x_2, \dots, x_m)$ is the unique solution of the system (1.1), i. e. $x_i = t_i^{(m)}$, $(i = 1, 2, \dots, m)$.

The system (1.1) defines a sequence of polynomials

$P_{\sigma,m}(x) = \prod_{i=1}^{m} (x - t_i^{(m)})$ called σ- orthogonal polynomials (Ghizzetti-Ossicini [2]).

In this paper we are concerned with asymptotic behaviour of the solution of (1.1) when only one of the integers s_i goes to infinity. The following results are obtained:

(1.4) $m \in N$, $\lim_{s_1 \to \infty} t_1^{(m)} = 0$; (This result was proved in [9])

(1.5) $m \geqslant 2$, $\lim_{s_i \to \infty} t_i^{(m)} = 1$ $(i = 2, 3, \dots, m)$,

$$\lim_{s_m \to \infty} t_m^{(m)} = 0,$$

$$\lim_{s_m \to \infty} t_i^{(m)} = -1 \quad (i = 1, 2, \dots, m-1).$$

(1.6) $m \geqslant 3$ $\lim_{s_j \to \infty} t_j^{(m)} = 0$ $(j = 2, 3, \dots, m-1)$

$$\lim_{s_j \to \infty} t_i^{(m)} = -1 \quad (i = 1, 2, \dots, j-1)$$

$$\lim_{s_j \to \infty} t_k^{(m)} = 1 \quad (k = j+1, j+2, \dots, m).$$

2. Case, when $s_2 = s_3 = \ldots = s_m = 0$.

In the case $s_2 = s_3 = \ldots = s_m = 0$ the system (1.1) is equivalent to

$$(2.1)\begin{cases} \int_{-1}^{1} p(x)(x-t)^{2s} \prod_{i=1}^{m} (x-t_i)\, dx = 0, \\ \int_{-1}^{1} p(x)(x-t_1)^{2s} \prod_{i=1}^{m} (x-t_i) \prod_{h=m-k}^{m} (x-t_h)\, dx = 0, \quad (k=0, 1, \ldots, m-2), \end{cases}$$

with $s = s_1$.

The equation of (2 1) corresponding to $k=m-3$ (For $m=2$ the first equation of (2.1) must be considered) can be written as

$$(2\,2) \quad \int_{-1}^{1} p_m(x,s)\, (x-t_1)^{2s+1} (x-t_2)\, dx = 0,$$

where $p_m(x,s) = p(x) \prod_{i=3}^{m} (x-t_i)^2$.

We observe that condition $-1 < t_i < 1$, $(i=3,4,\ldots m)$, implies $0 \leqslant p_m(x,s) \leqslant 2^{2(m-2)} p(x)$ uniformly for $x \in [-1,1]$.
It follows

$$(2.3) \quad \int_{-1}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+2} dx + \left(t_1^{(m)} - t_2^{(m)}\right) \int_{-1}^{1} p_m(x,s)(x-t_1^{(m)})^{2s+1} dx = 0$$

which implies $\int_{-1}^{1} p_m(x,s)(x-t_1^{(m)})^{2s+1} dx > 0$, because $t_1^{(m)} < t_2^{(m)}$.

Now we state:

Theorem I $\lim\limits_{s\to\infty} t_i^{(m)} = 1$, $i=2,3,\ldots m)$

Proof. Let us first prove $\lim\limits_{s\to\infty} t_2^{(m)} = 1$.

Form (2.3) we have

$$(2.4)\quad t_2^{(m)} - t_1^{(m)} = \frac{\int_{-1}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+2}\,dx}{\int_{-1}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+1}\,dx} \geq \frac{\int_{t_1^{(m)}}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+2}\,dx}{\int_{t_1^{(m)}}^{1} p_m(x,s)(x-t_1^{(m)})^{2s+1}\,dx}$$

Hölder's inequality gives

$$(2.5)\quad \int_{t_1^{(m)}}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+1}\,dx \leqslant \left(\int_{t_1^{(m)}}^{1} p_m(x,s)(x-t_1^{(m)})^{2s+2}\,dx \right)^{(2s+1)/(2s+2)} \cdot$$

$$\left(\int_{t_1^{(m)}}^{1} p_m(x,s)dx \right)^{1/(2s+2)} .$$

Hence

$$(2.6)\quad t_2^{(m)} - t_1^{(m)} \geq \frac{\left(\int_{t_1^{(m)}}^{1} p_m\,(x,s)\,(x-t_1^{(m)})^{2s+2}\,dx \right)^{1/(2s+2)}}{\left(\int_{t_1^{(m)}}^{1} p_m(x,s)\,dx \right)^{1/(2s+2)}} .$$

It is obvious $\lim_{s\to\infty} \left(\int_{t_1^{(m)}}^{1} p_m\,(x,s)\,dx \right)^{1/(2s+2)} = 1$.

Now we must show

$$(2.7)\qquad \lim_{s\to\infty} \left(\int_{t_1^{(m)}}^{1} p_m\,(x,s)\,(x-t_1^{(m)})^{2s+2}\,dx \right)^{1/(2s+2)} = 1 .$$

We remark: i) $\forall\, x \in [\,t_1^{(m)}, 1\,] \Rightarrow x-t_1^{(m)} \leqslant 1-t_1^{(m)}$; ii) since $\lim_{s\to\infty} t_1^{(m)} = 0$, for every $\varepsilon \geqslant 0$ there exists $s_\varepsilon \in N$ such that for every $s > s_\varepsilon\,(s \in N)$, $|\,t_1^{(m)}\,| \lhd \frac{\epsilon}{2}$ *holds.*

Then, for $s > s_{\varepsilon}$,

$$\left(\int_{t_1^{(m)}}^{1} p_m(x,s)\,(x-t_1^{(m)})^{1/(2s+2)}\,dx\right)^{1/(2s+2)} \geqslant \left(\int_{1-\epsilon/2}^{1} p_m(x,s)\,(x-t_1^{(m)})^{2s+2}\,dx\right)^{1/(2s+2)} \geqslant$$

$$\geqslant (1-\epsilon/2-t_1^{(m)})\left(\int_{1-\epsilon/2}^{1} p_m(x,s)\,dx\right)^{1/(2s+2)} \geqslant (1-\epsilon)\left(\int_{1-\epsilon/2}^{1} p_m(x,s)\,dx\right)^{1/(2s+2)}.$$

Hence

$$\text{(2.8)} \quad 1-\epsilon < \lim_{s\to\infty}{}' \,(t_2^{(m)} - t_1^{(m)}) = \lim_{s\to\infty}{}' \, t_2^{(m)} \leqslant \lim_{s\to\infty}{}'' \, t_2^{(m)} \leqslant 1,$$

then

$$\text{(2.9)} \quad \lim_{s\to\infty} t_2^{(m)} = 1 \quad \forall\, m \in N,\ m \geqslant 2$$

Because of $-1 < t_1^{(m)} < \ldots < t_m^{(m)} < 1$, $\forall\, s \in N$, and (2.9) theorem I is proved.

3. General case.

The system (1.1) is equivalent to

$$\begin{cases} \displaystyle\int_{-1}^{1} p(x) \prod_{i=1}^{m} (x-t_i)^{2s_i+1}\,dx = 0 \\ \displaystyle\int_{-1}^{1} p(x) \prod_{i=1}^{m} (x-t_i)^{2s_i+1} \prod_{h=m-k}^{m} (x-t_h)\,dx = 0 \quad (k=0, 1,\ldots,m-2). \end{cases}$$

If $m=2$ the first equation of (3.1) has to be considered and

$$P_2^*(x,s_1) = P(x)\,(x-t_2)^{2s_2+2}.$$

Therefore the equation of (3.1) corresponding to $k=m-3$ can be written as

$$\int_{-1}^{1} P_m^*(x,s_1)\,(x-t_1)^{2s_1+1}\,(x-t_2)\,dx = 0,$$

where $\overset{*}{p}_m(x, s_1) = p(x)\,(x-t_2)^{2s_2} \prod_{i=3}^{m} (x-t_i)^{2s_i+2}$.

We observe that condition $-1 \leqslant t_i < 1$, $(i=2,3,..,m)$, implies

$$0 \leqslant \overset{*}{p}_m(x, s_1) \leqslant 2^H p(x), \text{ where } H = 2\left(\sum_{i=2}^{m} s_i + m - 2\right)$$

uniformly for $x \in [-1,1]$.

We first state

Theorem II $\lim\limits_{s_1\to\infty} t_i^{(m)} = 1$, $(i=2, 3,...,m)$.

The proof is the same as that of Theorem I.

Now we can prove

Theorem III $\lim t_m^{(m)} = 0$; $\lim t_i^{(m)} = -1$, $(i=1,2,...,m-1)$

Proof: Let us write (1.1) in this way

$$\int_{-1}^{1} p(-x)\, x^k \prod_{i=1}^{m} (x + t_i)^{2s_i+1}\, dx = 0, \ (k=0,1,..,m-1).$$

This system has unique solution

$(-t_m^{(m)}, -t_{m-1}^{(m)}, ..., -t_1^{(m)})$ with

$-1 \leqslant -t_m^{(m)} < -t_{m-1}^{(m)} < ... < -t_1^{(m)} < 1$, which verifies,

because of Theorem II, $\lim(-t_m^{(m)}) = 0$, $\lim(-t_i^{(m)}) = 1$,

$(i=1,2,...,m-1)$, then Theorem III is proved.

Let us consider $m \geqslant 3$. When $s_j \to \infty$, $(2 \leqslant j \leqslant m-1)$, and s_i are fixed $(i=1,2,..,m;\ i \neq j)$, we state

Theorem IV $\lim\limits_{s_j\to\infty} t_j^{(m)} = 0$.

Proof. We shall show

$$\lim_{s_j\to\infty}{}' t_j^{(m)} = \lim_{s_j\to\infty}{}'' t_j^{(m)} = 0,\ 2 \leqslant j \leqslant m\text{-}1.$$

If $\lim''_{s_j\to\infty} t_j^{(m)} = T_j > 0$, there would be a subsequence $\{ t_{j,\nu_h}^{(m)} \}$ converging to T_j, then $t_{j,\nu_h}^{(m)} > 0$ definitely.

The last equation of (3.1) can be replaced by

$$(3.2)\quad \int_{-1}^{1} (p(x) \prod_{\substack{i=1\\ i\neq j}}^{m} (x\text{-}t_i)^{2s_i+2} (x\text{-}t_j)^{2s_j+1}\, dx = 0 .$$

Consider the sequence $(t_{1,\nu_h}^{(m)}, t_{2,\nu_h}^{(m)},\ldots,t_{m,\nu_h}^{(m)})$, $(h=1,2..)$; let

$$\bar{P}_m(x,s_j) = p(x) \prod_{\substack{i=1\\ i\neq j}}^{m} (x\text{-}t_{i,\nu_h}^{(m)})^{2s_i+2},$$

$0 \leqslant \bar{P}_m(x,s_j) \leqslant 2^L p(x)$ where $L = 2(\sum_{i=1}^{m} s_i + m\text{-}1)$,
$\forall\ x \in [-1,1]$ uniformly.

From (3 2) we have

$$0 = \int_{-1}^{-1+\frac{1}{2}t_{j,\nu_h}^{(m)}} \bar{P}_m(x,s_j)(x\text{-}t_{j,\nu_h}^{(m)})^{2\nu_h+1}\, dx + \int_{-1+\frac{1}{2}t_{j,\nu_h}^{(m)}}^{t_{j,\nu_h}^{(m)}} \bar{P}_m(x,s_j)(x\text{-}t_{j,\nu_h}^{(m)})^{2\nu_h+1}\, dx +$$

$$+ \int_{t_{j,\nu_h}^{(m)}}^{1} \bar{P}_m(x,s_j)(x\text{-}t_{j,\nu_h}^{(m)})^{2\nu_h+1}\, dx .$$

When $h \to \infty$, the first integral $\to -\infty$, the second one is negative, the third one is bounded. Hence $T_j \leqslant 0$.

By the same method $lim' \, t_j^{(m)} \geqslant 0$ can be proved; then theorem IV is proved.

Finally we can prove

Theorem V (i) $\lim_{S_j \to \infty} t_i^{(m)} = -1$, $(i=1,2,\ldots,j-1)$;

(ii) $\lim_{S_j \to \infty} t_k^{(m)} = 1$, $(k=j+1, j+2,\ldots,m)$.

Proof. The system (1.1) can be written

$$\begin{cases} \int_{-1}^{1} p(x) \prod_{i=1}^{m} (x-t_i)^{2s_i+1} \, dx = 0 \\ \int_{-1}^{1} p(x) \prod_{i=1}^{m} (x-t_i)^{2s_i+1} \prod_{h=1}^{k} (x-t_h) \, dx = 0 \quad (k=1,2,\ldots,j-1), \\ \int_{-1}^{1} p(x) \prod_{i=1}^{m} (x-t_i)^{2s_i+1} \prod_{h=1}^{j-1} (x-t_h) \prod_{l=m-r}^{m} (x-t_l) \, dx = 0 \end{cases}$$

$(r = 0,1,\ldots,m-j-1)$.

Let us write the equation corresponding to $r=m-j-2$ in the form:

$$\int_{-1}^{1} P_m(x,s_j)(x-t_j)^{2s_j+1}(x-t_{j+1}) \, dx = 0,$$

where $P_m(x,s_j) = \frac{p(x)}{(x-t_{j+1})^2} \prod_{\substack{i=1 \\ i \neq j}}^{m} (x-t_i)^{2s_i+2}$

Because of Theorem II $\lim_{S_j \to \infty} t_{j+1}^{(m)} = 1$, hence (ii) follows from (1.2).

By the same method and because of Theorem III, (i) can be proved.

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