Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

BINOMIAL ANALOGUES OF THE CLASS OF ADDITION THEOREMS OF SRIVASTAVA, LAVOIE AND TREMBLAY

By

R. C. Singh Chandel and Shashi Agrawal

Department of Mathematics, D. V. Postgraduate College
Orai-285001, U. P., India

(Received: June 4, 1987; Revised: January 16, 1989)

ABSTRACT

In the present paper, some binomial analogues of the class of addition theorems of Srivastava, Lavoie and Tremblay [8] are derived and a number of interesting applications of main results are given.

1. INTRODUCTION

Srivastava [6] presented certain interesting classes of generating functios in the forms:

$$\sum_{n=0}^{\infty} f_n^{\alpha+\lambda n}(x+ny)\frac{t^n}{n!}$$

and

$$\sum_{n=0}^{\infty} g_n^{(\alpha_1+\lambda_1 n,\ldots,\alpha_s+\lambda_s n)}(x_1+ny_1,\ldots,x_s+ny_s)\frac{t^n}{n!}$$

where

$$\left\{ f_n^{(\alpha)}(x) \right\}_{n=0}^{\infty} \text{ and } \left\{ g_n^{(\alpha_1,\ldots,\alpha_s)}(x_1,\ldots,x_s)\right\}_{n=0}^{\infty}$$

are general one and many-parameter sequences of functions. Further, Srivastava, Lavoie and Tremblay [8] derived some general addition formulas for analogous sequences of functions and gave a number of interesting applications of the main results. Recently, Chandel and Sahgal [3] also gave a number of interesting applications and extensions of the addition theorems of srivatava, Lavoie and Tremblay [8]. In the present paper, we derive some binomial analogues of the class of addition theorems of Srivastava, Lavoie and Tremblay [8] and give a number of interesting applications of main results.

2. Some Theorems. In this section, we establish following theorems:

Theorem 1. *Let the functions $B(z)$ and $z^{-1} C(z)$ be analytic in the neighbourhood of the origin, and assume (for the sake of simplicity) that*

$$(2.1) \quad B(0) = C'(0) = 1.$$

Define the sequence of functions $\left\{ G_n^{(\alpha, \beta)}(x) \right\}_{n=0}^{\infty}$ *by means of*

$$(2.2) \quad [B(z)]^{\alpha} [1-xC(z)]^{-\beta} = \sum_{n=0}^{\infty} G_n^{(\alpha, \beta)}(x) \frac{z^n}{n!},$$

where α, β and x are arbitrary complex numbers independent of z. Then for arbitrary parameters λ and μ independent of z,

$$(2.3) \quad G_n^{(\alpha+\lambda\gamma, \beta+\mu\gamma)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{n+\gamma}{k+\gamma}\right) G_k^{(\alpha-\lambda k,\ \beta-\mu k)}(x)$$

$$G_{n-k}^{(\gamma(k+\lambda),\ \mu(k+\gamma))}(x),$$

provided that $Re(\gamma) > 0$.

Theorem 2. *Let the functions $A(z)$, $B(z)$, $z^{-1} C(z)$ and $z^{-1} D(z)$ be analytic about the origin such that*

$$(2.4) \quad A(0) = B(0) = C'(0) = D'(0) = 1,$$

and define the sequence of functions $\left\{ G_n^{(\alpha, \beta, \gamma, \lambda, \mu)}(x, y, u, v) \right\}_{n=0}^{\infty}$ *by means of*

$$(2.5) \quad \sum_{n=0}^{\infty} G_n^{(\alpha, \beta, \gamma, \lambda, \mu)}(x, y, u, v) \frac{z^n}{n!}$$

$$= [A(z)]^{\alpha} [B(z)]^{\beta} [\Omega(z)]^{\gamma} [1-x\, C(z)]^{-\lambda} [1-y\, D(z)]^{-\mu},$$

where $\alpha, \beta, \gamma, \lambda, \mu, u, v, x$ and y are independent of z and (for convenience)

(2.6) $\Omega(z) = \left[1 + \dfrac{zuB'(z)}{B(z)} + \dfrac{zvxC'(z)}{1-xC(z)} \right].$

Then

(2.7) $G_n^{(\alpha_1+\alpha_2,\ \beta_1+\beta_2,\ \gamma_1+\gamma_2+1,\ \lambda_1+\lambda_2,\ \mu_1+\mu_2)}(x,y,u,v)$

$= \sum_{k=0}^{n} \binom{n}{k} G_k^{(\alpha_1,\ \beta_1-uk,\ \gamma_1+1,\ \lambda_1-vk,\ \mu_1)}(x,y,u\ v)$

$G_{n-k}^{(\alpha_2,\ \beta_2+uk,\ \gamma_2+1,\ \lambda_2+vk,\ \mu_2)}(x,y,u,v)\ .$

This theorem may be regarded as a generalization of the Theorem 1. More generally, we have

Theorem 3. *Let the function $A(z)$, $B(z)$ and $z^{-1}\ C_i(z)$ be analytic about the origin such that*

(2.8) $A(0) = B(0) = C'_i(0) = 1,\quad i=1,\dots,m$

and define the sequence of functions

$$\left\{ G_n^{(\alpha,\beta,\gamma,\lambda_1,\dots,\lambda_m)}(x_1,\dots,x_m,\ u,v_1,\dots,v_m) \right\}_{n=0}^{\infty}$$ *by means of*

(2.9) $\sum_{n=0}^{\infty} G_n^{(\alpha,\beta,\gamma,\lambda_1,\dots,\lambda_m)}(x_1,\dots,x_m,\ u,v_1,\dots,v_m)\dfrac{z^n}{n!}$

$= [A(z)]^{\alpha}\ [B(z)]^{\beta}\ [\Omega(z)]^{\gamma}\ [\ 1-x_1\ C_1(z)\ ]^{-\lambda_1} \dots [\ 1-x_m\ C_m(z)\ ]^{-\lambda_m}$

where $\lambda_1,\dots,\lambda_m$, $x_1,\dots,x_m$, $v_1,\dots,v_m$, u, α, β *and* γ *are all arbitrary complex numbers independent of* z, *and (for convenience)*

(2.10) $\Omega(z) = 1 + z\left[\dfrac{uB'(z)}{B(z)} + \dfrac{v_1 x_1 C'_1(z)}{1-x_1C_1(z)} + \dots + \dfrac{v_m x_m C'_m(z)}{1-x_mC_m(z)} \right].$

Then

(2.11) $G_n^{(\alpha+\alpha',\ \beta+\beta',\ \gamma+\gamma'+1,\ \lambda_1+\mu_1,\dots,\lambda_m+\mu_m)}(x_1,\dots,x_m,\ u,\ v_1,\dots,v_m)$

$$= \sum_{k=0}^{n} \binom{n}{k} \; G_k^{(\alpha,\, \beta-ku, \gamma+1, \lambda_1-kv_1, \ldots, \lambda_m-k\mu_m)}(x_1,\ldots,x_m,\, u, v_1,\ldots,v_m)$$

$$G_{n-k}^{(\alpha',\, \beta'+ku,\, \gamma'+1,\, \mu_1+kv_1, \ldots, \mu_m+kv_m)}(x_1,\ldots,x_m,\, u,\, v_1,\ldots,v_m).$$

3. **Proof of Theorem** 3. Following the techniques applied for tne proof of addition formula due to Srivastava, Lavoie and Tremblay [8, p. 440, (1.7)], we can prove Theorem 3. For highlighting its connections with the earlier results, we may also refer to Srivastava and Manocha [9, Chapter 7, Problem 30].

4. **Applications of Theorem** 1. Chandel and Chandel [:] have given the following interesting special case for their polynomials

$$(i\text{-}pz^q)^{-c}\left[1 - \frac{xz}{(1-pz^q)^r}\right]^{-b} = \sum_{n=0}^{\infty} g_n^{(b,c)}(x, p, q, r)\, z^n.$$

Thus for $\alpha = c$, $\beta = b$, $B(z) = (1\text{-}pz^q)^{-1}$ and $C(z) = \dfrac{z}{(1-z^q)^r}$, the Theorem 1 gives

$$(4.1) \quad g_n^{(b+\lambda\gamma,\; c+\mu\gamma)}(x, p, q, r)$$

$$= \sum_{k=0}^{n} \binom{n+\gamma}{k+\gamma} g_k^{(b-\lambda k,\, c-\mu k)}(x, p, q, r)\; g_{n-k}^{(\lambda(k+\gamma),\; \mu(k+\gamma))}(x, p, q, r).$$

For $s = 1$, the special case of the polynomials due to Chandel and Yadava [4], is defined as

$$[a_0 + a_1x_1 z + \ldots + a_mx_m z^m]^p \left[[1 - \frac{r^r x_1 z}{(a_0+a_1x_1z+\ldots+a_mx_m z^m)^r}\right]^{-q}$$

$$= \sum_{n=0}^{\infty} E^{a_0,\ldots,a_m}_{n,m,\, p,q,r,\, 1}\begin{bmatrix} x_1 \;,\; x_i \\ \vdots \\ x_m \end{bmatrix} z^m,$$

so that in the Theorem **1**, taking $\alpha = p$, $\beta = q$,
$B(z) = a_0 + a_1x_1z + \ldots + a_mx_mz^m$ and

$$C(z) = \frac{r^r z}{(a_0 + a_1 x_1 z + \ldots + a_m x_m z^m)^r},$$

we get

$$(4.2) \quad E^{a_0,\ldots,a_m}_{n,\, m,\, p+\lambda\gamma,\, q+\mu\gamma,\, r,\, 1} \begin{bmatrix} x_1 , x_i \\ \vdots \\ x_m \end{bmatrix} =$$

$$= \sum_{k=0}^{n} \binom{n+\gamma}{k+\gamma} E^{a_0,\ldots,a_m}_{k,\, p-\lambda k,\, q-\mu k,\, r,\, 1} \begin{bmatrix} x_1 , x_i \\ \vdots \\ x_m \end{bmatrix}$$

$$\cdot\, E^{a_0,\ldots,a_m}_{n-k,\lambda(k+\gamma),\, \mu(k+\gamma),\, r,\, 1} \begin{bmatrix} x_1 , x_i \\ \vdots \\ x_m \end{bmatrix}, \quad Re(\gamma) > 0.$$

Chandel [1,2] introduced and studied generalized Stirling numbers whose generating relation is given by

$$\sum_{n=0}^{\infty} S^{(\alpha,\, k)}(n,m,r)\frac{z^n}{n!} = \frac{(-1)^m}{m!}\,[1-(k-1)z]^{-\alpha/(k-1)}$$

$$\left[1-\{1-(k-1)z\}^{-r/(k-1)}\right]^m, \quad k \neq 1 .$$

Thus an appeal to the Theorem 1, with $x=1$, $\beta = -m$,
$B(z) = [1-(k-1)z]^{-1/(k-1)}$ and $C(z) = [1-(k-1)z]^{-r/(k-1)}$

gives

$$(4.3) \quad S^{(\alpha+\lambda\gamma,\, k)}(n, m+\mu\gamma, r)$$

$$= \sum_{i=0}^{n} \binom{n}{i} \binom{n+\gamma}{i+\gamma} \frac{(m-\mu i)!\,[\mu(i+\gamma)]!}{(m-\mu\gamma)!}$$

$$S^{(\alpha-\lambda i,\, k)}(i, m-\mu i, r)\; S^{(\lambda(i+\gamma),\, k)}(n-i, \mu(i+\gamma), r), \quad Re(\gamma)>0$$

For $r=q=1$, $s=0$ the relation (19) due to Srivastava and Buschman [7] reduces to

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-\alpha)_n z^n}{n!}\; {}_{p+1}F_p \begin{bmatrix} -n,\ \beta,\ \triangle(p-1,\ -\alpha+n); & \dfrac{(p-1)^{p-1}}{p^p}x \\ \triangle(p,\ -\alpha); & \end{bmatrix}$$

$$= (1+z)^{\alpha} \left[1 - \frac{xz}{(1+z)^p}\right]^{-\beta}.$$

Therefore taking $B(z) = 1+z$, $C(z) = \dfrac{z}{(+z)^p}$ and making an appeal to the Theorem 1, we obtain

$$(4.4) \quad (\alpha+\lambda\gamma)_n \; {}_{p+1}F_p \left[\begin{matrix} -n, \beta+\mu\gamma \Delta(p-1, a+\lambda\gamma+n); \\ \Delta(p, \alpha+\lambda\gamma); \end{matrix} \; x\right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{n+\gamma}{k+\gamma}\right) (\alpha-\lambda k)_k \, (\lambda(k+\gamma)_{n-k}$$

$${}_{p+1}F_p \left[\begin{matrix} -k, \beta-\mu k, \Delta(p \; 1, a-\lambda k+k); \\ \Delta(p \; \alpha-\lambda k); \end{matrix} \; x\right]$$

$${}_{p+1}F_p \left[\begin{matrix} -(n-k), \mu'k+\gamma). \; \Delta(p-1, \lambda(k+\gamma) + n \; k); \\ \Delta(p. \lambda \; k+\gamma)); \end{matrix} \; x\right].$$

Acknowledgments

The authors are thankful, to Professor H. M. Srivastava for his suggestions regarding the presentation of the paper.

REFERENCES

[1] R. C. S. Chandel, A further note on the class of polynomials $T_n^{(\alpha,k)}(x, r, p)$, *Indian J. Math.* **16** (1974), 39 48

[2] R. C. S. Chandel, Generalized Stirling numbers and polynomials, *Publ. Inst. Math. (Beograd) (N. S.)* **22** (36) (1977), 43 48.

[3] R. C. Singh Chandel and S. Sahgal, Further applications and extensions of the addition theorems of Srivastava, Lavoie and Tremblay," *Indian J. Pure Appl. Math.*, **18** (1987), 830-834.

[4] R. C. S. Chandel and H. C. Yadava, Some generating functions for certain polynomial systems in several variables, *Proc. Nat. Acad. Sci. India Sect. A*, **51** (1981), 133-138.

[5] R. S. Chandel and R. C. S. Chandel, A new class of polynomials and the polynomials related to them, *Rev. Tècn. Fac. Ingr. Univ. Zulia*, 7 (1985), 63-67.

[6] H. M. Srivastava, Some generalizations of Carlitz's tneorem, *Pacific J. Math.* 85 (1979), 471-477.

[7] H. M. Srivastava and R. G. Buschman, Some polynomials defined by generating relations, *Trans. Amer. Math Soc.*, 205 (1975), 360-370.

[8] H. M. Srivastava, J.-L. Lavoie and R. Tremblay, A class of addition theorems. *Canad Math. Bull.*, 26 (1983), 438-445.

[9] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions, Halsted Press*, John Wiley and Sons, New York, 1984.