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(Dedicated to the memory of Professor K. L. Singh)

UNIFIED PRESENTATION OF TWO GENERAL SEQUENCES OF FUNCTIONS

By

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ABSTRACT

In the present paper, an unified study of two general sequences of functions defined by Bhargava [1] and Chandel-Agrawal [4], has been made through operational techniques.

1. INTRODUCTION

To unify the study two general classes of functions due to Chandel and Bhargava [2] and Singh [5], Bhargava [1] introduced a sequence of functions defined by :

$$(1.1) \quad G_n\,[a, k, p; g(x), h(x)] = e^{-p\,g(x)}\, T^n_{a\,k} \left\{ h^n(x),\ e^{p\,g(x)} \right\},$$

where $T_{a,k} = x^k\,(a + xD)$; $h(x)$, $g(x)$ are suitable functions of x and a, k, p are independent of x.

Recently, to unify the study of two general classes of functions of Chandel-Bhargava [2] and Chandel-Agrawal [3], Chandel and Agarwal [4] introduced a sequence of functions

defined by :

$$(1.2) \quad G_n^{(a,b,k)}\,(h, g(x)) = [\,1 - h\,g(x)]^{b}\; T^n_{a\,k} \left\{ (1 - h\,g(x))^{-b} \right\},$$

where a, b, h, k are independent of x and $g(x)$ is a suitable differentiable function of x.

The motivation of this paper is to unify the study of two general saquences of functions defined by (1.1) and (1.2), by introducing the sequence of functions

$$\left\{ R_n^{(a,b,k,p)} (h(x), g(x)) \;/\; n=0,1,2, \ldots \right\}$$

defined by Rodrigues' formula

(1.3) $$R_n^{(a,b,k,p)} (h(x), g(x)) = [1-pg(x)]^b T_{a,k}^n \left\{ (h(x))^n [1-pg(x)]^{-b} \right\},$$

where a, b, k, p are independent of x and $h(x)$, $g(x)$ are suitable differentiable functions of x.

Particularly replacing p by p/b and $b \to \infty$, (1.3) reduces to (1.1). For $h(x)$ independent of x and $p=h$, (1.3) reduces to (1.2).

2. Operational formulas. Consider

$$[1-pg(x)]^b T_{a,k}^n \left\{ (h(x))^n [1-p\,g(x)]^{-b} f \right]$$

$$= [1-p\,g(x)]^b \sum_{s=0}^{n} \binom{n}{s} T_{a,k}^{n-s} \left\{ (h(x))^n [1-p\,g(x)]^{-b} \right\} T_{0,k}^{s} \{f\}$$

$$= [1-p\,g(x)]^b \sum_{s=0}^{n} \binom{n}{s} T_{0,k}^{n-s} \left\{ [(h(x))^{n/(n-s)}]^{n-s} [1-pg(x)]^{-b} \right\} T_{0,k}^{s} \{f\}$$

Therefore

(2.1) $$[1-p\,g(x)]^b T_{a,k}^n \left\{ (h(x))^n [1-pg(x)]^{-b} f \right\}$$

$$= \sum_{s=0}^{n} \binom{n}{s} R_{n-s}^{(a,b,k,p)} \left((h(x))^{n/(n-s)}, g(x) \right) T_{0,k}^{s} \{f\}.$$

Again considering

$$[1-pg(x)]^b \; T^n_{a,k} \left\{ (h(x))^n \; [1-pg(x))^{-b} \; f \right\}$$

$$= [1-pg(x)]^b \; T^{n-1}_{a,k} \left\{ (ax^k + x^{k+1}D) \; [(h(x))^n \; (1-pg(x))^{-b} \; f \,] \right\}$$

$$= [1-pg(x)]^b \; T^{n-1}_{a,k} \left\{ (h(x))^n \; (1-pg(x))^{-b} \right.$$

$$\left. \left[x^k \, (a+xD) + x^{k+1}\left(\frac{nh'(x)}{h(x)} + \frac{pbg'(x)}{1-pg(x)} \right)\right] \right\} f$$

$$= [1-pg(x)]^b \; ((h(x)^n \; [1-pg(x)]^{-b} \; \left[T_{a,k} + x^{k+1}\left(\frac{nh'(x)}{h(x)} + \frac{pbg'(x)}{1-pg(x)} \right)\right]^n f,$$

we obtain

$$(2.2 \quad [1-pg(x)]^b \; T^n_{a,k} \; \{(h(x))^n \; (1-pg(x))^{-b} f \}$$

$$= (h(x))^n \left[T_{a,k} + x^{k+1} \left(\frac{nh'(x)}{h(x)} + \frac{pbg'(x)}{1-pg(x)} \right)\right]^n f .$$

From (2.1) and (2.2), we derive

$$(2.3) \quad ((h(x))^n \left[T_{a,k} + x^{k+1} \left(\frac{nh'(x)}{h(x)} + \frac{pb\,g'(x)}{1-p\,g(x)} \right)\right]^n f$$

$$= \sum_{s=0}^{n} \binom{n}{s} R^{(a,b,k,p)}_{n-s} \Big((h(x))^{n/(n-s)}, \; g(x) \Big) \; T^s_{0,k} \; \{ f \},$$

which particularly, for $f(x) = 1$, further gives

$$(2.4) \left[T_{a,k} + x^{k+1} \left(\frac{nh'(x)}{h(x)} + \frac{pb\,g'(x)}{1-pg(x)} \right)\right]^n \{ 1 \}$$

$$= (h(x))^{-n} \; R^{(a,b,k,p)}_n \; (h(x)^{n/(n-s)}, g(x)).$$

Next consider

$$T^n_{a,k} \left\{ (h(x))^n \; [1-pg(x)^{-b} \; f \right\}$$

$$= T_{a,k}^{n-1} \left\{ (h(x))^{n-1} (1-pg(x))^{-b} \left[h(x)\, T_{a,k} + x^{k+1}\left(nh'(x) + \frac{pbg'(x)h(x)}{1-pg(x)} \right) \right] f \right\}$$

$$= T_{a,k}^{n-2} \left\{ (h(x))^{n-2} (1-pg(x))^{-b} \left[h'(x)\, T_{a,k} + x^{k+1} \left((n-1)h'(x) + \frac{pbh(x)g'(x)}{1-pg(x)} \right) \right] \right.$$

$$\left. \left[h(x)\, T_{a,k} + x^{k+1}\left(nh'(x) + \frac{pbh(x)g'(x)}{1-pg(x)} \right) \right] f \right\}$$

$$= h(x)^{n-n} (1-pg(x))^{-b} \left[h(x)\, T_{a,k} + x^{k+1}\left(n-(n-1)h'(x) + \frac{bpg'(x)\, h(x)}{1-pg(x)} \right) \right]$$

$$\ldots \left[h(x)\, T_{a,k} + x^{k+1} \left(nh'(x) + \frac{pb\, h(x)\, g'(x)}{1-pg(x)} \right) \right] f.$$

Therefore

$$(2.5) \qquad T_{a,k}^{n} \left\{ (h(x))^{n} [1-pg(x)]^{-b} f \right\}$$

$$= [1-pg(x)]^{-b} \prod_{j=0}^{n-1} \left[h\, T_{a,k} + x^{k+1} \left((n-j)h'(x) + \frac{pbh(x)\, g'(x)}{1-p\, g(x)} \right) \right] f.$$

Making an appeal to (2.1) the above result (2.5) gives

$$(2.6) \qquad \sum_{s=0}^{n} \binom{n}{s} R_{n-s}^{(a,b,k,p)} \left((h(x))^{n/(n-s)}, g(x) \right) T_{0,k}^{s} \{f\}$$

$$= \prod_{j=0}^{n-1} \left[h(x)\, T_{a,k} + x^{k+1} \left\{ (n-j)\, h'(x) + \frac{pb\, h(x) g'(x)}{1-pg(x)} \right\} \right] \{f\},$$

which particularly for $f(x) = 1$, further gives

$$(2.7) \qquad \prod_{j=0}^{n-1} \left[h(x)\, T_{a,k} + x^{k+1} \left\{ (n-j)\, h'(x) + \frac{pbh(x)\, g'(x)}{1-pg(x)} \right\} \right] \{1\}$$

$$= R_{n}^{(a,b,k,p)} (h(x), g(x)).$$

Again consider

$$T^{n}_{a,k}\left\{(h(x))^{n}[1-pg(x)]^{-b}f\right\}$$

$$=T^{n-1}_{a,k}\left\{(h(x))^{n-1}[1-pg(x)]^{-b}\left[x^{k+1}\left(nh'(x)+\frac{pbh(x)g'(x)}{1-pg(x)}\right)+h(x)T_{a,k}\right]f\right\}$$

$$=T^{n-1}_{a,k}\left\{(h(x))^{n-(m+1)}[1-pg(x)]^{-b}\left[x^{k+1}\left\{(n-m-1+(m+1)(h(x))^{m}h'(x)\right.\right.\right.$$

$$\left.\left.\left.+\frac{(h(x))^{m+1}pbg'(x)}{1-pg(x)}\right\}+(h(x))^{m+1}T_{a,k}\right]f\right\}$$

$$=(h(x))^{-nm}[1-pg(x)]^{-b}\prod_{j=1}^{n}\left[x^{k+1}\left\{(n-(m+1)+(m+1))(h(x))^{m}h'(x)\right.\right.$$

$$\left.\left.+\frac{(h(x))^{m+1}pbg'(x)}{1-pg(x)}\right\}+(h(x))^{m+1}T_{a,k}\right]f.$$

Thus

$$(2.8)\qquad T^{n}_{a,k}\left\{(h(x))^{n}[1-pg(x)]^{-b}f\right\}$$

$$=(h(x))^{-nm}[1-pg(x)]^{-b}\prod_{j=0}^{n-1}\left[x^{k+1}\left\{(n-(m+1)j)(h(x))^{m}h'(x)+\right.\right.$$

$$\left.\left.\frac{(h(x))^{m+1}pbg'(x)}{1-pg(x)}\right\}+(h(x))^{m+1}T_{a,k}\right]f,$$

which by making an appeal to (2.1), gives

$$(2.9)\qquad (h(x))^{nm}\sum_{s=0}^{n}\binom{n}{s}R^{(a,b,k,p)}_{n-s}\left((h(x))^{n/(n-s)},g(x)\right)T^{s}_{0,k}\{f\}$$

$$=\prod_{j=0}^{n-1}\left[(h(x))^{m+1}T_{a,k}+x^{k+1}\{n-(m+1)j\}(h(x))^{m}h'(x)\right.$$

$$\left.+\frac{(h(x))^{m+1}pbg'(x)}{1-pg(x)}\right]f.$$

Particularly for $f(x) = 1$, from above result, we derive

$$(2.10)\quad \prod_{j=0}^{n-1} \left[(h(x))^{m+1} T_{a,k} + x^{k+1} \left\{ \frac{(h(x))^{m+1} pbg'(x)}{1-pg(x)} \right.\right.$$

$$\left.\left. + (n-(m+1)j)(h(x))^m h'(x) \right\} \right] \{1\} = (h(x))^{nm} R_n^{(a,b,k,p)} (h(x), g(x)).$$

Further consider

$$T_{a,k}^{n} \left\{ (h(x))^n [1-pg(x)]^{-b} f \right\}$$

$$= T_{a,k}^{n-1} \left\{ (h(x))^n [1-pg(x)]^{-b} x^{-m} \left[x^{m+k+1} \left(\frac{nh'(x)}{h(x)} + \frac{pbg'(x)}{1-pg(x)} \right) + x^m T_{a,k} \right] f \right\}$$

$$= T_{a,k}^{n-2} \left\{ (h(x))^n x^{-2m} [1-pg(x)]^{-b} \left[x^{m+k+1} \left(\frac{nh'(x)-mh(x)/x}{h(x)} + \frac{pbg'(x)}{1-pg(x)} \right) \right.\right.$$

$$\left.\left. + x^m T_{a,k} \right] \left[x^{m+k+1} \left(n \frac{h'(x)}{h(x)} + \frac{pbg'(x)}{1-pg(x)} \right) + x^m T_{a,k} \right] f \right\}.$$

Finally, we derive

$$(2.11)\quad T_{a,k}^{n} \left\{ (h(x))^n [1-pg(x)]^{-b} f \right\}$$

$$= (h(x)) x^{-nm} [1-pg(x)]^{-b} \prod_{j=1}^{n} \left[x^{m+k+1} \left(\frac{nh'(x)}{h(x)} + \frac{pbg'(x)}{1-pg(x)} \right) \right.$$

$$\left. + x^m T_{a,k} - m(j-1) x^{m+k} \right] f,$$

which by making an appeal to (2.1), gives

$$(2.12)\quad (h(x))^n x^{-nm} \prod_{j=1}^{n} \left[x^{m+k+1} \left(\frac{nh'(x)}{h(x)} + \frac{pbg'(x)}{1-pg(x)} \right) + x^m T_{a,k} \right.$$

$$\left. - m(j-1) x^{m+k} \right] f$$

$$= \sum_{s=0}^{n} \binom{n}{s} R_{n-s}^{(a,b,k,p)} \left((h(x))^{n/(n-s)}, g(x) \right) T_{0,k}^{s} \{f\}$$

Particularly for $f(x)=1$, the above result further gives

$$(2.13)\quad (h(x))^n x^{-nm} \prod_{j=1}^{n}\Big[x^{m+k+1}\left(\frac{nh'(x)}{h(x)}+\frac{pbg'(x)}{1-pg(x)}\right) + x^m T_{a,k} - m(j-1)\,x^{m+k}\Big]\{1\}$$

$$= R_n^{(a,b,k,p)}(h(x),g(x)).$$

3. **Other relations** Consider

$$T_{a,k}^{s}\left\{ R_n^{(a,b,k,p)}(h(x),g(x))\right\}$$

$$= T_{a,k}^{s}\Big[[1-pg(x)]^{-b}\, T_{a,k}^{n}\left\{ (h(x))^n [1-pg(x)]^{-b}\right\}\Big]$$

$$= \sum_{j=0}^{s}\binom{s}{j} T_{a,k}^{n+s-j}\left\{(h(x))^n[1-pg(x)]^{-b}\right\} T_{0,k}^{j}\left\{[1-pg(x)]^{b}\right\}.$$

Therefore, we derive

$$(3.1)\quad T_{a,k}^{s} R_n^{(a,b,k,p)}(h(x),g(x))$$

$$= \sum_{j=0}^{s}\binom{s}{j} R_{n+s-j}^{(a,b,k,p)}\left((h(x))^{n/(n+s-j)},\, g(x)\right) R_j^{(0,-b,k,p)}(1,g(x)).$$

Particularly for $s=1$, we further derive

$$(3.2)\quad T_{a,k}\left\{ R_n^{(a,b,k,p)}(h(x),g(x))\right\}$$

$$= R_{n+1}^{(a,b,k,p)}((h(x))^{n/(n+1)},g(x)) + R_n^{(a,b,k,p)}(h(x),g(x))\, R_1^{(0,-b,k,p)}(1,g(x)).$$

Since

$$R_1^{(0,-b,k,p)}(1,g(x)) = -\frac{bpg'(x)\,x^{k+1}}{1-pg(x)},$$

Therefore, (3.2) further gives

$$(3.3)\qquad \left(T_{a,k} + \frac{bpg'(x)\, x^{k+1}}{1\text{-}pg(x)} \right) R_n^{(a,b,k,p)} \left(h(x), g(x) \right)$$

$$= R_{n+1}^{(a,b,k,p)} \left(h(x))^{n/(n+1)}, g(x) \right).$$

By iteration we derive

$$(3.4)\quad \left(T_{a,k} + \frac{bpg'(x)\, x^{k+1}}{1\text{-}pg(x)} \right)^r \left\{ R^{(a,b,k,p)} \left(h(x), g(x) \right) \right\}$$

$$= R_{n+r}^{(a,b,k,p)} \left((h(x))^{n/(n+r)}, g(x) \right).$$

For brevity, we write

$$(3.5)\quad T_{a,k} + \frac{bpg'(x)\, x^{k+1}}{1+pg(x)} = \circledast .$$

Therefore, we have

$$(3.6)\qquad \circledast^r \left\{ R_n^{(a,b,k,p)} \left(h(x), g(x) \right) \right\}$$

$$= R_{n+r}^{(a,b,k,p)} \left((h(x))^{n/(n+r)}, g(x) \right) .$$

By making an appeal to (3.1) anb (3.6), we derive

$$(3.7)\qquad T_{a,k}^{s} \left\{ R_n^{(a,b,k,p)} \left(h(x), g(x) \right) \right\}$$

$$= \sum_{j=0}^{s} \binom{s}{j} R_j^{(0,-b,k,p)} \left(1, g(x) \right) \circledast^{s-j} \left\{ R_n^{(a,b,k,p)} \left(h(x), g(x) \right) \right\}$$

Further an appeal to (3.6) and the operational formula due to Chandel and Agrawal [4, (4.6)]

$$e^{t\circledast} \{ f(x) \} = \frac{[1\text{-}pg(x)]^b}{(1\text{-}x^k kt)^{a/k}} \left[1\text{-}pg \left(\frac{x}{(1\text{-}x^k kt)^{1/k}} \right) \right]^{-b}$$

$$f \left(\frac{x}{(1\text{-}x^k k\, t)^{1/k}} \right).$$

shows that

$$(3.8)\quad \frac{[1-pg(x)]^{b}}{[1-x^{k}k\,t\,]^{a/k}}\left[1-pg\left(\frac{x}{(1-x^{k}k\,t\,)^{1/k}}\right)\right]^{-b}$$

$$R_{n}^{(a,b,k,p)}\left[h\left(\frac{x}{(1-x^{k}kt)^{1/k}}\right),g\left(\frac{x}{(1-x^{k}kt)^{1/k}}\right)\right]$$

$$=\sum_{r=0}^{\infty}R_{n+r}^{(a,b,k,p)}\left((h(x))^{n/(n+r)},g(x)\right)\frac{t^{r}}{r!}.$$

Replace t by t/x^{k}, we obtain

$$(3.9)\quad \frac{[1-pg(x)]^{b}}{(1-t)^{a/k}}\left[1-pg\left(\frac{x}{(1-t)^{1/k}}\right)\right]^{-b}$$

$$R_{n}^{(a,b,k,p)}\left[h\left(\frac{x}{(1-t)^{1/k}}\right),g\left(\frac{x}{(1-t)^{1/k}}\right)\right]$$

$$=\sum_{r=0}^{\infty}R_{n+r}^{(a,b,k,p)}\left((h(x))^{n/(n+r)},g(x)\right)\frac{t^{r}}{r!\,x^{kr}\,k^{r}}.$$

Particularly for $n=0$, we derive

$$(3.10)\quad \frac{[1-pg(x)]^{b}}{(1-t)^{a/k}}\left[1-pg\left(\frac{x}{(1-t)^{1/k}}\right)\right]^{-b}$$

$$=\sum_{r=0}^{\infty}R_{r}^{(a,b,k,p)}\left(1,g(x)\right)\frac{t^{r}}{r!\,k^{r}\,x^{kr}}.$$

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