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(Dedicated to the memory of Professor K. L. Singh)

ON SOME NEW RESULTS INVOLVING DOUBLE AND TRIPLE HYPERGEOMETRIC FUNCTIONS

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ABSTRACT

The aim of this paper is to establish some new generating relations for certain hypergeometric functions of two and three variables.

1. INTRODUCTION

If we use the notation $(a)_n = a(a+1) \ldots (a+n-1)$, $(a)_0 = 1$, where a is arbitrary and n a positive integer, then the needed hypergeometric functions of two and three variables are defined as follows:

$$F_E(\alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p} (\beta_1)_m (\beta_2)_{n+p} x^m y^n z^p}{m! \, n! \, p! \, (\gamma_1)_m (\gamma_2)_n (\gamma_3)_p}$$

$$|x| < r, |y| < s, |z| < t, \qquad \ldots (1.1)$$

$$F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p} (\beta_1)_m (\beta_2)_n (\beta_3)_p x^m y^n z^p}{m! \, n! \, p! \, (\gamma_1)_m (\gamma_2)_{n+p}}$$

$$|x| < 1, |y| < 1, |z| < 1, \qquad \ldots (1.2)$$

$$F_A(\alpha, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+n+p} (\beta_1)_m (\beta_2)_n (\beta_3)_p x^m y^n x^p}{m! \, n! \, p! \, (\gamma_1)_m (\gamma_2)_n (\gamma_3)_p}$$

$$|x| + |y| + |z| < 1, \qquad \ldots (1.3)$$

$H_A(\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p} (\beta)_{m+n} (\beta')_{n+p} x^m y^n z^p}{(\gamma)_m (\gamma')_{n+p} m! n! p!}$$

$$|x| < r, |y| < s, |z| < t, \quad r+s+t = 1 + st, \qquad \ldots (1.4)$$

$G_B(\alpha, \beta_1, \beta_2, \beta_3; \gamma; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m} (\beta_1)_m (\beta_2)_n (\beta_3)_p x^m y^n z^p}{(\gamma)_{n+p-m} m! n! p!}$$

$$|x| < 1, |y| < 1, |z| < 1, \qquad \ldots (1.5)$$

${}^3G_B^{(1)}(\alpha, \beta_1, \beta_2; \gamma; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m} (\beta_1)_m (\beta_2)_n x^m y^n z^p}{(\gamma)_{n+p-m} m! n! p!} . \qquad \ldots (1.6)$$

$\Gamma_1(\alpha, \beta, \beta', x, y)$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\beta)_{n-m} (\beta')_{m-n} x^m y^n}{m! n!} . \qquad \ldots (1.7)$$

$H_2(\alpha, \beta, \gamma, \delta, \epsilon, x, y)$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n} (\beta)_m (\gamma)_n (\delta)_n x^m y^n}{(\epsilon)_m m! n!} . \qquad \ldots (1.8)$$

$H_3(\alpha, \beta, \gamma, x\, y)$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n} (\beta)_n x^m y^n}{(\gamma)_{m+n} m! n!} . \qquad \ldots (1.9)$$

(See, for details, [1], [2], [3], [5] and [8].)

In the present investigation we also require the following relations:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n t^n}{n!} F_1(-n,\mu,\nu;\alpha;x,y) = (1-t)^{-\lambda} F_1(\lambda,\mu,\nu;\alpha;\frac{xt}{t-1},\frac{yt}{t-1})$$

$$\max\left[\left|\frac{xt}{t-1}\right|, \left|\frac{yt}{t-1}\right|, |t|\right\} < 1, \qquad \ldots (1,10)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n t^n}{n!} F_2(-n,\mu,\nu;\alpha,\beta;x,y) = (1-t)^{-\lambda} F_2(\lambda,\mu,\nu;\alpha,\beta;\frac{xt}{t-1},\frac{yt}{t-1}),$$

$$\max\left\{\left|\frac{xt}{t-1}\right| + \left|\frac{yt}{t-1}\right|, |t|\right\} < 1, \qquad \ldots (1.11)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n t^n}{n!} F_4(-n,\mu;\alpha,\beta;x,y) = (1-t)^{-\lambda} F_4(\lambda,\mu,\alpha,\beta;\frac{xt}{t-1},\frac{yt}{t-1}),$$

$$\max\left\{|xt/(t-1)|^{1/2} + |yt/(t-1)|^{1/2}, |t|\right\} < 1, \qquad \ldots (1.12)$$

$$\sum_{n=0}^{\infty} (u-1)^n P_n^{(\alpha-n,\beta+n)}\left(\frac{1+u}{1-u}\right) {}_2F_1(-n,\delta-\gamma,\delta,t)$$

$$= \frac{\Gamma(\delta)(\delta+\alpha-\gamma)}{\Gamma(\delta+\alpha)\Gamma(\delta-\gamma)} t^{\alpha}(1-u+ut)^{-\beta} {}_2F_1\left(\beta,\gamma,\beta+\alpha, \frac{-ut}{1-u+ut}\right), \qquad \ldots (1.13)$$

where F_1, F_2, F_4 are Appell's double hypergeometric functions [2,p.224 (6), (7), (9)] and (1.10), (1.11), (1.12) are known results given by Srivastava [6, p. 86, (4.1), (4.2), (4.3)].

2. GENERATING RELATIONS

We prove here the following formulae:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} L_1(\alpha,1-\beta,\lambda+n,x,y) F_4(-n,\mu;\nu,\epsilon;u,v)t^n$$

$$= (1-t)^{-\lambda} \sum_{q=0}^{\infty} \frac{(1-\beta)_q (y(t-1))^q}{(1-\lambda)_q\, q!} \cdot F_E(\lambda-q,\lambda-q,\alpha,\mu,\mu;\beta-q,\nu,\epsilon;\frac{x}{t-1},\frac{ut}{t-1},\frac{vt}{t-1}),$$

$$\ldots (2.1)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} H_2(\lambda+n,\alpha,\beta,\gamma,\delta,x,y)\, F_1(-n,\mu,\nu;\epsilon;u,v)t^n$$

$$= (1-t)^{-\lambda} \sum_{q=0}^{\infty} \frac{(\beta)_q(\gamma)_q(y(t-1))^q}{q!\,(1-\lambda)_q},$$

$$F_G\left(\lambda-q,\ \lambda-q,\ \lambda-q,\alpha,\mu,\nu;\delta,\epsilon,\epsilon;\ \frac{x}{1-t},\frac{ut}{t-1},\frac{vt}{t-1}\right); \quad \ldots (2.2)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} H_3(\lambda+n,\ \beta,\ \gamma,\ x,y,\ F_2(-n,\mu,\nu;\alpha,\epsilon;u,v,)t^n$$

$$= (1-t)^{-\lambda} \sum_{p=0}^{\infty} \frac{(\lambda)_{2p}}{(\gamma)_p\, p!} \left(\frac{x}{(1-t,^2}\right)^p$$

$$F_A\left(\lambda+2p,\beta,\mu,\nu;\gamma+p,\alpha,\epsilon;\ \frac{\lambda}{1-t},\ \frac{ut}{t-1},\ \frac{vt}{t-1}\right). \quad \ldots (2.3)$$

To prove (2.1), consider:

$$\phi = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \Gamma_1(\alpha,1-\beta,\ \lambda+n,\ x,y)\ F_4\ (-n,\mu;\nu,\epsilon;\ u,v)t^n$$

Express Γ_1 in series form as given in (1.7), employ

$$(\lambda)_n\,(\lambda+n)_{p-q} = (\lambda)_{n+p-q} = (\lambda)_{p-q}\,(\lambda+p-q)_n, \quad \ldots (2.4)$$

and apply relation (1.12); we find that

$$\phi = \sum_{p,q=0}^{\infty} \frac{(\alpha)_p\,(1-\beta)_{q-p}\,(\lambda)_{p-q}\,x^p,y^q}{p!\,q!}$$

$$F_4(\lambda+p-q,\mu;\nu,\epsilon;\ \left(\frac{ut}{t-1},\ \frac{vt}{t-1}\right).\ (1-t)^{-(\lambda+p-q)}.$$

Writing F_4 in series form, we have

$$\phi = (1-t)^{-\lambda} \sum_{q=0}^{\infty} \frac{(\lambda)_{-q}\,(1-\beta)_q\,(y(1-t))^q}{q!}$$

$$\sum_{r,s,p=0}^{\infty} \frac{(\lambda-q)_{p+r+s}\,(\mu)_{r+s}\,(\alpha)_p}{(\beta-q)_p\,(\nu)_r\,(\epsilon)_s\,r!\,p!\,s!} \left(\frac{x}{t-1}\right)^p \left(\frac{ut}{t-1}\right)^r \left(\frac{vt}{t-1}\right)^s$$

which in the light of (1.1) provides (2.1).

The derivations of formulas (2.2), and (2.3) would run parallel to what we have obtained above.

3. GENERATING RELATIONS

In this section the following generating relations have been established:

$$(1-z)^{\beta}(1-y)^{\gamma} H_A\left(\beta, \gamma, \beta'; \delta+\alpha, \beta'; \frac{-xt(1-z)(1-y)}{(1-x+xt)}, y, z\right)$$

$$= \frac{\Gamma(\delta+\alpha)\,\Gamma(\delta-\gamma)}{\Gamma(\delta)\,\Gamma(\delta+\alpha-\gamma)} t^{-\alpha}(1-x+xt)^{\beta} \sum_{n=0}^{\infty} \frac{n!}{(\delta)_n}(x-1)^n P_n^{(\alpha-n,\ \beta+n)}\left(\frac{1+x}{1-x}\right)$$

$$\cdot P_n^{(\delta-1,\ -\gamma-n)}(1-2t), \qquad \ldots (3.1)$$

$$(1-z)^{\beta_3} G_B(\alpha, \beta_1, \beta_2, \beta_3; \alpha; x, x(1-y)/2, z)$$

$$= \sum_{m=0}^{\infty} (-x)^m P_m^{(-m-\beta_1, \beta_2+\beta_1-1)}(y). \qquad \ldots (3.2)$$

$${}_3G_B^{(1)}(\alpha, \beta, \gamma; \alpha; x, x(1-y)/2, z)$$

$$= \exp(x) \sum_{m=0}^{\infty} (-x)^m P_m^{(-\beta-m,\ \gamma+\beta-1)}(y). \qquad \ldots (3.3)$$

To prove (3.1), let us consider:

$$T = (1-z)^{\beta}(1-y)^{\gamma} H_A\left(\beta, \gamma, \beta'; \delta+\alpha, \beta'; \frac{-xt(1-z)(1-y)}{(1-x+xt)}, y, z\right)$$

and express H_A in series form, applying the result

$$(1-z)^{-a} = \sum_{i=0}^{\infty} \frac{(a)_i z^i}{i!}, \qquad \ldots (3.4)$$

we get

$$T = {}_2F_1\left(\beta, \gamma, \delta+\alpha; \frac{-xt}{(1-x+xt)}\right),$$

which in the light of (1.13) provides (3.10).

The proofs of the results (3.2) and (3.3) are similar to that of (3.1).

4. PARTICULAR CASES

In (2.1) putting $\nu = 0$ and simplifying we obtain:

$$\sum_{p,q=0}^{\infty} \frac{(\alpha)_p (\lambda)_{p-q} (1-\beta)_{q-p} \left(\frac{x}{1-t}\right)^p (y(1-t))^q}{p!\, q!} \; {}_2F_1\left(\lambda+p-q, \mu; \nu; \frac{ut}{t-1}\right)$$

$$= \sum_{q=0}^{\infty} \frac{(1-\beta)_q (y(t-1))^q}{q!\,(1-\lambda)_q} F_2\left(\lambda-q, \alpha, \mu; \beta-q, \nu; \frac{x}{t-1}, \frac{ut}{t-1}\right). \qquad \ldots (4.1)$$

In (3.1) letting $z=0$ and simplifying, we get

$${}_2F_1\left(\beta, \nu, \delta+\alpha, \frac{-xt}{1-x+xt}\right) = \frac{\Gamma(\delta+\alpha)\Gamma(\delta-\nu)}{\Gamma(\delta)\Gamma(\delta+\alpha-\nu)} t^{-\alpha} (1-x+xt)^{\beta}$$

$$\sum_{n=0}^{\infty} \frac{(x-1)^n n!}{(\delta)_n} P_n^{(\alpha-n,\, \beta+n)}\left(\frac{1+x}{1-x}\right) . P_n^{(\delta-1,\, -\gamma-n)}(1-2t). \qquad \ldots (4.2)$$

In (3.2) taking $z=0$ replacing $1-y$ by $\frac{1-y}{\beta_2}$ and letting $\beta_2 \to \infty$ we obtain:

$$(1-x)^{-\beta_1} \sum_{n=0}^{\infty} \frac{\left[x\left(\frac{1-y}{2}\right)\right]^n}{n!} = \sum_{m=0}^{\infty} (-x)^m L_m^{(-\beta_1-m)}\left(\frac{1-y}{2}\right). \qquad \ldots (4.3)$$

In (3.3) changing $1-y$ by $\frac{1-y}{\gamma}$ and putting $\gamma \to \infty$, we obtain

$$exp\,(z)\,(1-x)^{-\beta} \sum_{n=0}^{\infty} \frac{\left[x\left(\frac{1-y}{2}\right)\right]^n}{n!} = exp(z) \sum_{m=0}^{\infty} (-x)^m L_m^{(-\beta-m)}\left(\frac{1-y}{2}\right),$$

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