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(Dedicated to the memory of Professor K. L. Singh)

AN EXPANSION FORMULA FOR MULTIVARIRBLE H-FUNCTION INVOLVING GENERALIZED LEGENDRE'S ASSOCIATED FUNCTION

By

R. K. Saxena and Anil Ramawat

Department of Mathematics and Statistics, Jay Narain Vyas University Jodhpur-342 001, Rajiasthan

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ABSTRACT

The authors establish a new expansion formula for multivariable H-function due to Srivastava and Panda [13] in terms of a series of products of the multivariable H-function and the generalized Legendre's associated function due to Meulenbeld [9]. A result given earlier by Anandani [2] follows as a special case.

1. INTRODUCTION

The multivariable H-function is defined by means of the following r-tuple contour integral [14, *p*, 251, *Eq*. (*c*. 1)]:

$$(1.1)\quad H[z_1,\dots,z_r] = H^{0,\,n:\,m_1,\,n_1;\dots;m_r,\,n_r}_{p,\,q:\,p_1,\,q_1;\dots;p_r,\,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix}(a_j;\alpha_j',\dots,\alpha_j^{(r)})_{1,p}: \\ (b_j;\beta_j',\dots\beta_j^{(r)})_{1,q}: \end{matrix}\begin{matrix}(c_j',\gamma_j')_{1,q_1};\dots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (d_j',\delta_j')_{1,q_1};\dots;(d_j^{(r)}\,\delta_j^{(r)})_{1,q_r}\end{matrix}\right]$$

$$= [1/(2\pi w)^r]\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\dots\phi_r(\xi_r)\;\Psi\;(\xi_1,\dots,\xi_r)z_1^{\xi_1}\dots z_r^{\xi_r}\,.\,d\xi_1\dots d\xi_r,$$

where $w = \sqrt{(-1)}$ and for the definition of the functions $\Psi(\dots)$ and

$\phi_i(\xi_i)$ $(i = 1,\dots,r)$ and as also the condition of existence of the multivariate H-function, we refer to [14, *p.* 251-253, Eqns. (*c*. 2)-(*c*.8)].

The various conditions of existence of the multivariate H-function are assumed to be satisfied for multivariate H-function occuring in this paper.

In this paper we evaluate an integral involving generalized associated Legendre function and the multivariate H-function due to Srivastava and Panda and apply it in deriving an expansion for the multivariate H-function in series of products of associated Legendre function and the multivariable H-function.

2. The Integral

The integral to be evaluated is

$$(2.1)\quad \int_{-1}^{1} (1-x)^{\rho-u/2} (1+x)^{\sigma+v/2} P_{k-(u-v)/2}^{u,v}(x)$$

$$\cdot\, H[(1-x)^{\alpha_1} (1-x)^{\beta_1} z_1,\dots,(1-x)^{\alpha_r} (1+x)^{\beta_r} z_r]\, dx$$

$$= 2^{\rho-u+v+\sigma+1} \sum_{t=0}^{\infty} \frac{(-k)_t\,(v-u+k+1)_t}{\Gamma(1-u+t)\; t!}$$

$$\cdot\, H^{0,\, n+2:m_1,\, n_1;\dots;m_r,\, n_r}_{p+2,\, q+1:p_1,\, q_1;\dots;p_r,\, q_r}\left[\begin{matrix} 2^{\alpha_1+\beta_1} z_1 \\ \vdots \\ 2^{\alpha_r+\beta_r} z_r \end{matrix}\,\middle|\, \begin{matrix} (-\sigma-v;\beta_1,\dots,\beta_r); \\ (b_j;\beta_j',\dots,\beta_j^{(r)})_{1,q}: \end{matrix}\right.$$

$$\begin{matrix} (u-\rho-t;\alpha_1,\dots,\alpha_r);\ (a_j;\alpha_j',\dots,\alpha_j^{(r)})_{1,p}\ :\ (c_j',\gamma_j')_{1,p_1};\dots; \\ (u-v-\rho-\sigma-t-1;\ \alpha_1+\beta_1,\dots,\alpha_r+\beta_r)\ :\ (d_j',\delta_j')_{1,q_1};\dots; \end{matrix}$$

$$\left.\begin{matrix} (c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{matrix}\right].$$

The integral (2. 1) is valid under the following set of conditions:

(i) $(\beta_i, \alpha_i) > 0$; $\forall i \in \{1,\ldots,r\}$; $k - \frac{u-v}{2}$ is a positive integer, k is an integer $\geqslant 0$;

(ii) $Re(\rho - u + \sum_{i=1}^{r} \alpha_i \frac{d_j^{(i)}}{\delta_j^{(i)}} > -1$; $Re(\sigma + v + \sum_{i=1}^{r} \beta_i \frac{d_j^{(i)}}{\delta_j^{(i)}}) > -1$;

$(j = 1,\ldots,m_i;\ i = 1,\ldots,r)$

and the conditions given in [14, *p.* 252-253; Eqns. (*c.* 4), (*c.* 5) and (*c.* 6)] are also satisfied.

Proof. On expressing the multivariable H-function in the integrand as a multiple Mellin-Barnes type integral (1.1) and inverting the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, the value of the integral

$$= (2\pi w)^{-r} \int_{L_1} \cdots \int_{L_r} \Psi(s_1,\ldots,s_r) \sum_{i=1}^{r} \left\{ \phi_i(s_i)\ z_i^{\xi_i} \right\}$$

$$\cdot \left\{ \int_{-1}^{1} (1-x)^{\rho - u/2 + \sum_{i=1}^{r} \alpha_i \xi_i} (1+x)^{\sigma + v/2 + \sum_{i=1}^{r} \beta_i \xi_i} \right.$$

$$\cdot P_{k-(u-v)/2}^{u,v}(x) dx\, d\xi_1 \ldots d\xi_r.$$

On evaluating the x-integral with the help of the integral [10, *p.* 343, *Eq.* (38)]:

$$(2.2) \int_{-1}^{1} (1-x)^{\rho} (1-x)^{\sigma} P_{k-(m-n)/2}^{m,\,n}(x) dx$$

$$= \frac{2^{\rho+\sigma-(m-n)/2}\ \Gamma(\rho - \frac{m}{2} + 1)\ \Gamma(\sigma + \frac{n}{2} + 1)}{\Gamma(1-m)\ \Gamma(\rho + \sigma - \frac{m-n}{2} + 2)}$$

$$\cdot\ {}_3F_2\left(-k,\ n-m+k+1,\ \rho - \frac{m}{2} + 1;\ 1-m,\ \rho - \sigma - \frac{m-n}{2} + 2;\ 1\right),$$

provided that $Re(\rho - \frac{m}{2}) > -1$, $Re(\sigma + \frac{n}{2}) > -1$; and interpreting the result with the help of (1.1), the integral (2.1) is established.

3. Expansion Theorem

Let the following conditions be satisfied:

(i) $\beta_1, \ldots, \beta_r > 0$; $\alpha_1, \ldots, \alpha_r \geqslant 0$ (or $\beta_1, \ldots, \beta_r \geqslant 0$; $\alpha_1, \ldots, \alpha_r > 0$);

(ii) n, p, q, m_i, n_i, p_i and q_i are integers such that

$0 \leqslant n \leqslant p$, $q \geqslant 0$, $1 \leqslant m_i \leqslant q_i$ and $0 \leqslant n_i \leqslant p_i$,

$\forall\, i \in \{1, \ldots, r\}$, and the conditions given by [14, *p.* 252–253, Eqns. (*c.* 4), (*c.* 5) and (*c.* 6)] also satisfied.

(iii) $Re(v) > -1$, $Re(u) > 1$, $Re(\rho - u + \sum_{i=1}^{r} \alpha_i \frac{d_j^{(i)}}{\delta_j^{(i)}}) > -1$;

$Re(\sigma + v + \sum_{i=1}^{r} \beta_i \frac{d_j^{(i)}}{\delta_j^{(i)}}) > -1$, $(j = 1, \ldots, m_i;\ i = 1, \ldots, r)$.

Then the following expansion formula holds:

$$(3.1)\quad (1-x)^{\rho-u/2}(1+x)^{\sigma+v/2}\, H[(1+x)^{\alpha_1}(1+x)^{\beta_1} \cdot z_1, \ldots, (1-x)^{\alpha_r}(1+x)^{\beta_r} z_r]$$

$$= 2^{\rho+\sigma} \sum_{N=0}^{\infty} \sum_{\mu=0}^{N} \frac{(2N-u+v+1)\,\Gamma(N-u+1)\,\Gamma(1+v-u+N+\mu)\,(-N)_\mu}{N!\,\mu!\,\Gamma(1+v+N)\,\Gamma(1-u+\mu)}$$

$$P^{u,v}_{N-(u-v)/2}(x) \cdot H^{0,\,n+2:\,m_1,\,n_1;\ldots;m_r,\,n_r}_{p+2,\,q+1:p_1,\,q_1;\ldots;p_r,\,q_r}\left[\begin{matrix} 2^{\alpha_1+\beta_1} z_1 \\ \vdots \\ 2^{\alpha_r+\beta_r} z_n \end{matrix}\right|$$

$$(-v-\sigma;\beta_1,\ldots,\beta_r);\ (u-\rho-\mu;\alpha_1,\ldots,\alpha_r),\ (a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,p}:(c_j',\gamma_j')_{1,p_1};$$

$$(b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,q};\ (-1-\rho-\sigma-\mu+u-v;\ \alpha_1+\beta_1,\ldots,\alpha_r+\beta_r):(d_j',\delta_j')_{1,q_1}$$

$$;\dots;\,(c_j^{(r)},\ \gamma_j^{(r)})_{1,\,p_r}\Big]$$
$$;\dots;\,(d_j^{(r)},\ \delta^{(r)})_{1,\,q_r}\Big].$$

Proof. Let

$$(3.\ 2)\quad f(x) = (1-x)^{\rho-u/2}\,(1+x)^{\sigma+v/2}\,H[\,(1-x)^{\alpha_1}\,(1+x)^{\beta_1}z_1;\dots;$$
$$(1-x)^{\alpha_r}\,(1+x)^{\beta_r}\,z_r\,]$$
$$= \sum_{N=0}^{\infty} C_N\, P^{u,\,v}_{N-(u-v)/2}(x)$$

Equation (3. 2) is valid since $f(x)$ is continuous and of bounded variation in the interval (-1, 1).

Now, multiplying both the sides of (3. 2) by $P^{u,\,v}_{k-(u-v)/2}(x)$ and integrating with respect to x from -1 to 1; evaluating the L. H. S. with the help of (2. 1) and on the R. H. S. interchanging the order of summation, using [4, *p*. 176, Eq. (75)] and then applying orthogonality property of the generalized Legendre's associated functions [10, *p*. 340, Eq. (26) and Eq. (27)]:

$$(3.\ 3)\quad \int_{-1}^{1} P^{u,\,v}_{k-(u-v)/2}(x)\; P^{u,\,v}_{N-(u-v)/2}(x)\,dx$$

$$= \begin{cases} 0, \ if\ k \neq N \\ \dfrac{2^{u-v+1}\,k!\,\Gamma(k+v+1)}{(2k-u+v+1)\,\Gamma(k-u+1)\,\Gamma(k-u+v+1)}\ ;\ if\ k = N \end{cases}$$

provided that $Re(u) < 1$, $Re(v) > -1$; we obtain

$$(3.4)\quad C_k = \frac{2^{\rho+\sigma}\,(2k-u+v+1)\,\Gamma(k-u+1)}{k!\,\Gamma(k+v+1)} \sum_{\mu=0}^{k} \frac{(-k)_\mu\,\Gamma(1+k+v-u+\mu)}{\mu!\,\Gamma(1-u+\mu)}$$

$$.\,H^{0,\ n+2:\ m_1,\ n_1;\ \ ;m_r,\ n_r}_{p+2,\ q+1:\ p_1,\ q_1;\ \cdot\ ;p_r,\ q_r}\left[\begin{matrix}2^{(\alpha_1+\beta_1)}\ z_1\\ 2^{(\alpha_r+\beta_r)}\ z_r\end{matrix}\middle|\begin{matrix}(-\nu-\sigma;\beta_1,\ldots,\beta_r);\\ (b_j;\ \beta_j',\ldots,\beta_j^{(r)})_{1,\ q};\end{matrix}\right.$$

$$(u-\rho-\mu;\ \alpha_1,\ldots,\alpha_r);\ (a_j,\ \alpha_j',\ldots,\alpha_j^{(r)})_{1,\ p}:$$

$$(-1-\rho-\sigma-\mu+u\ \ \nu;\ \alpha_1+\beta_1,\ldots,\alpha_r+\beta_r):$$

$$\left.\begin{matrix}(c_j',\ \gamma_j')_{1,p_1};\ldots;(c_j^{(r)},\ \gamma_j^{(r)})_{1,p_r}\\ (d_j',\ \delta_j')_{1,q_1};\ldots;(d_j^{(r)},\ \delta_j^{(r)})_{1,q_r}\end{matrix}\right].$$

Now on substituting the value of C_k in (3. 2), the result follows.

4. A Special Case

For $n=p=0$, $q=0$, the multivariate H-function breaks up into a product of r H-functions and consequently, (3. 1) reduces to

$$(4.\ 1)\quad (1-x)^{\rho-m/2}(1+x)^{\sigma+n/2}\prod_{i=1}^{r}\left\{H^{m_i,\ n_i}_{p_i,\ q_i}\left[(1-x)^{\alpha_i}(1+x)^{\beta_i}z_i\middle|\begin{matrix}(c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i}\\ (d_j^{(i)},\ \delta_j^{(i)})_{1,\ q_i}\end{matrix}\right]\right.$$

$$=2^{\rho+\sigma}\sum_{N=0}^{\infty}\sum_{\mu=0}^{N}\frac{(2N-u+\nu+1)\,\Gamma(N-u+1)\,\Gamma(1+\nu-u+N+\mu)(-N)_\mu}{n!\ \mu!\ \Gamma(1+\nu+N)\ \Gamma(1-u+\mu)}$$

$$.\,P^{u,\ \nu}_{N-(u-\nu)/2}(x)\,H^{0,\ 2:m_1,\ n\ ;\ldots;m_r,\ n_r}_{2,\ 1:\ p_1,\ q_1;\ldots;p_r,\ q_r}\left[\begin{matrix}2^{(\alpha_1+\beta_1)}\ z_1\\ \vdots\\ 2^{(\alpha_r+\beta_r)}\ z_r\end{matrix}\middle|\begin{matrix}(-\nu-\sigma;\\ (-1-\rho-\sigma-\mu\end{matrix}\right.$$

$$\begin{matrix}\ldots;\beta_1,\ldots,\beta_r)\ ;\ (u-\rho-\mu;\ \alpha_1,\ldots,\alpha_r)\\ +u-\nu;\ \alpha_1+\beta_1,\ldots,\alpha_r+\beta_r)\end{matrix}$$

$$\left.\begin{matrix}:(c_j',\ \gamma_j')_{1,\ p_1};\ldots;\ (c_j^{(r)},\ \gamma_j^{(r)})_{1,\ p_r}\\ :(d_j',\ \delta_j')_{1,\ q_1};\ldots;\ (d_j^{(r)},\ \delta_j^{(r)})_{1,\ q_r}\end{matrix}\right]$$

For $r=1$, (4. 2) gives rise to the result due to Anandani [2].

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