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(Dedicated to the memory of Professor K. L. Singh)

A GENERAL FIXED POINT THEOREM FOR EXPANSION MAPPINGS

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1. During past fifteen years fixed and common fixed point theorems were established for certain expansion mappings ([1], [2], [3]). In the present paper, we shall prove the following more general theorem which includes the above mentioned results as special cases :

Theorem. Let f, g be surjective selfmaps of a complete metric space (X, d). Suppose there exists a function p, q, r satisfying

(1. 1) $\inf_{x,yX} \{q(x, y) + r(x, y)\} > 1$

(1. 2) $\inf_{x,yX} \{1 - p(x, y)\, q(x, y) + r(x, y)\} > 0$

(1. 3) $\sup_{x,yX} \{p(x, y) + q(x, y)\} < 1$ *and*

(1. 4)
$$d(fx, gy) \geqslant p(x, y) \frac{d(x, fx)\, d(y, fy)}{d(x, y)} + q(x, y) \frac{d(y, gy)\,[1 + d(x, f(x))]}{[1 + d(x, y)]} + r(x, y)\, d(x, y)$$

for all x, y in X, $x \neq y$. Then f and g have common fixed point in X.

Proof. Let x_0 be in X. Since f is surjective, there exists a point x_1 in $f^{-1}x_0$, similarly g is surjective, there exists a point x_2 in $g^{-1}x_1$. Continuing we obtain a sequence $\{x_n\}$ with x_{2n+1} in $f^{-1}x_{2n}$, x_{2n+2} in $g^{-1}x_{2n+1}$.

Suppose $x_{2n+1} = x_{2n}$ for some n. If $x_{2n+1} \neq x_{2n+2}$, then from (1. 4) we write

$$(x_{2n+1}, x_{2n}) = d(gx_{2n+2}, fx_{2n+1}) = d(fx_{2n+1}, gx_{2n+2})$$

$$\geqslant p(x, y) \frac{d(x_{2n+1}, x_{2n})\, d(x_{2n+2}, x_{2n+1})}{d(x_{2n+1}, x_{2n+2})}$$

$$+ q(x, y) \frac{d(x_{2n+2}, x_{2n+1})\, [1 + d(x_{2n+1}, x_{2n})]}{1 + d(x_{2n+1}, x_{2n+2})}$$

$$+ r(x, y)\, d(x_{2n+1}, x_{2n+2}).$$

Thus

$$0 \geqslant (q + r)\, d(x_{2n+1}, x_{2n+2})$$

where q and r are evaluated at $(x, y) = (x_{2n+1}, x_{2n+2})$.

By (1. 2), $d(x_{2n+1}, x_{2n+2}) = 0$.

The condition $x_{2n+1} = x_{2n} = fx_{2n+1}$ implies that x_{2n+1} is a fixed point of f. Also $x_{2n+2} = x_{2n+1} = gx_{2n+2}$ implies x_{2n+2} is fixed point of g. Similarly $x_{2n+2} = x_{2n+1}$ implies $x_{2n+2} = gx_{2n+2}$ and $fx_{2n+1} = x_{2n+1}$ leads to x_{2n+1} as a common fixed point of f and g.

Again from (1. 4), we write

$$d(x_{2n+1}, x_{2n+2}) = d(gx_{2n+2}, fx_{2n+3}) = d(fx_{2n+3}, gx_{2n+2})$$

$$\geqslant p(x, y) \frac{d(x_{2n+3}, x_{2n+2})\, d(x_{2n+2}, x_{2n+1})}{d(x_{2n+3}, x_{2n+2})}$$

$$+ q(x, y) \frac{d(x_{2n+2}, x_{2n+1})\, [1 + d(x_{2n+3}, x_{2n+2})]}{[1 + d(x_{2n+3}, x_{2n+2})]}$$

$$+ r(x, y)\, d(x_{2n+2}, x_{2n+2}).$$

Therefore, $(1-p-q)\, d(x_{2n+1}, x_{2n+2}) \geqslant rd(x_{2n+2}, x_{2n+3})$,

where p, q, r are evaluated at (x_{2n+3}, x_{2n+2}).

Thus $\{x_n\}$ is a Cauchy sequence, hence is convergent to some x in X. Without loss of generality we may assume that $x_n = x$ for infinitely many n otherewise f and g have a common fixed point. If there exists an infinite number of integers n such that $x_{2n} \neq x$,

then for $y \in g^{-1}x$,

$$d(x_{2n}, x) = d(fx_{2n+1}, gy)$$

$$\geqslant p \frac{d(x_{2n+1}, x_{2n})\, d(y, gy)}{d(x_{2n+1}, y)}$$

$$+ q \frac{d(y, gy)\,[1 + d(x_{2n+1}, x_{2n})]}{[1+d(x_{2n+\cdot}, y)}$$

$$+ rd(x_{2n+1}, y)$$

where p, q, r are evaluated at (x_{2n+1}, y). Thus

$$d(x_{2n}, x) \geqslant (q+r) \min \{dx, y), d(x_{2n+1}, y)\}.$$

Therefore by (1. 1), $x = y$.

Thus if $x_{2n+1} \neq x$ for large n, then $x_{2n} = fx_{2n+1} = x$.

As $n \to \infty$. x is a fixed point of f.

If $x_{2n+1} \neq x$ for imfinitely many n, define $f^{-1}x = z$, and if p, q, r are evaluated at (z, x_{2n+2}) then by (1.4),

$$d(x_{2n+1}, x) = d(gx_{2n+2}, fz) = d(fz, gx_{2n+2})$$

$$\geqslant p \frac{d(z, fz)\, d(x_{2n+2}, x_{2n+1})}{d(z, x_{2n+2})}$$

$$+ q \frac{d(x_{2n+2}, x_{2n+1})\,[1+d(z, fz)]}{[1+d(z, x_{2n+2})]}$$

$$+ r\, d(z, x_{2n+2})$$

$$\geqslant \inf_{n,y} (q+r) \min_{X} \{d(z, x), d(z, x_{2n+2})\}$$

As $n \to \infty$, therefore an appeal to (1.1) implies $x=z$.

This completes the proof of the theorem.

Remark 1. For $p=q=0$, $r=a$, our theorem reduces to the theorem of Rhoades [3].

Remark 2. For $p=q=0$, $r=a$ and $f=g$, our theorem reduces to the theorem Iseki, Li and Gao [2].

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