Jñānābha, Vol, 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

A GENERAL SOLUTION OF A PARTIAL DIFFERENTIAL EQUATION OCCURRING IN THE ORNSTEIN-UHLENBECK DIFFUSION PROCESS

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(Received: July 26, 1991; Revised: October 31, 1991)

ABSTRACT

A general solution of the backword equation occurring in the Ornstein- Uhlenbeck diffusion process is obtained.

1. INTRODUCTION

In this note, we present a general solution of the backward equation [3, p. 155, (20)]:

$$(1.1)\quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - x\,\frac{\partial u}{\partial x}\ (-\infty < x < \infty),$$

which uniquely determines the behaviour of the Ornstein-Uhlenbeck (O-U) diffusion process. As a praticular case of our solution (2.4) below, we obtain the following elementary solution given by Karlin and McGregor [2, p. 170]:

$$(1.2)\quad p(t;x,y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi}2^{n+1}\,n!}\ e^{-nt} H_n(x/\sqrt{2})\, H_n(y/\sqrt{2}),$$

where $H_n(x)$ denotes the Hermite polynomial of degree n in x.

The backward equation (1.1) also occurs in the study of the limiting behaviour of the many-server queue [3, pp. 154-155].

2. THE SOLUTION OF THE BACKWARD PARTIAL DIFFERENTIAL EQUATION.

Let us assume that (1.1) has a solution of the form:

(2.1) $u(x,t) = e^{-nt}\, y(x/\sqrt{2}),\ n=0,1,2,\dots$.

The substitution of (2.1) into (1.1) yields the differential equation:

(2.2) $y''(x/\sqrt{2}) - \sqrt{2}\, xy'(x,\sqrt{2}) + 2ny'x/\sqrt{2}) = 0,$

which is Hermite's equation [3, p. 54, (23)] with solution $y = H_n(x/\sqrt{2})$. Hence we conclude that to each eigenvalue given by (2.2), there corresponds the solution of (1.1), called an eigenfunction or eigenstate given by

(2.3) $u_n = e^{-nt}\, H_n(x/\sqrt{2}),\ n=0,1,2,\dots$.

On using the principle of superposition, the solution of (1.1) takes the form:

(2.4) $u(x,t) = \sum_{n=0}^{\infty} C_n\, e^{-nt} H_n(x/\sqrt{2}).$

Putting t=0 in (2.4) we have

(2.5) $u(x,0) = \sum_{n=0}^{\infty} C_n\ H_n(x/2).$

Multiplying both sides of (2.5) by $e^{-x^2/2}\, H_m(x/\sqrt{2})$ and integrating with respect to x from $-\infty$ to ∞, and using the orthonality property of the Hermite polynomials [3, p. 54, (17)] after setting $x/\sqrt{2}$ for x, we obtain the Fourier-Hermite coefficients:

(2.6) $C_n = \dfrac{1}{2^n\, n!(2\pi)^{1/2}} \displaystyle\int_{-\infty}^{\infty} e^{-x^2/2}\ H_n(x/\sqrt{2}) u(x)\ dx.$

3. THE PARTICULAR SOLUTION

In (2.6), putting

$$u(x) = \frac{e^{y^2/4}}{2(2\pi)^{1/2}\, i^n}\quad e^{x^2/4 + iyx/2},$$

and using the following integral [1, p. 174, 7 (a)] (with $t=x/\sqrt{2}$ and $x=y/\sqrt{2}$) :

$$(3.1) \quad \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{-x^2/4 + iyx/2} H_n(x/\sqrt{2})\, dx=\sqrt{2}i^n e^{-y^2/4} H_n(y/\sqrt{2})$$

we get

$$(3.2) \quad C_n= \frac{1}{\sqrt{\pi}2^{n+1}n!} H_n(y/2).$$

Subtituting the value of C_n from (3.2) into (2.4), we obtain the elementary solution (1.2) given by Karlin and McGregor.

Remark 1. On evaluating the integral in (2.5) for a particular value of $u(x)$, the values of C_0 $C_1, C_2, \ldots$ for that value of $u(x)$ can be obtained. This exercise may be useful for computation.

Remark 2. In a future communication we propose to obtain the values of the Fourier-Hermite coefficients C_n for $u(x,0)$ as Meijer's G-function [3, pp. 37–39], which is a generalization of almost all special function appearing in applied mathematics, physical sciences, and statistics.

Acknowledgements

The author expresses his gratitude to Professor H. M. Srivastava for some useful suggestions for revising the paper.

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