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(Dedicated to the memory of Professor K. L. Singh)

AN EXTENSION OF BROUWER FIXED POINT THEOREM FOR CONTRACTIVE MAPS

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ABSTRACT

This note attempts to extend two recent fixed point theorems for contractive maps of Bridges, Richman, Julian and Mines [*J. Math. Anal, Appl.* **165** (1992), No. 2, 438–456].

1. INTRODUCTION

Let (X,d) be a metric space and $S : X \to X$. Then S is contractive (also called shrinking) if $d(Sx, Sy) < d(x, y)$ for all distinct points x, y of X. It is well known that (see [1], [6]) a contractive map on a compact metric space has a unique fixed point. In view of this classsical fixed point theorem of Edelstein [2], Smart [6, *p.* 39] raised the following question: Does every contractive map of the closed unit ball in a Banach space have a fixed point? This question has been negatively answered by *V.* Totik [7]. Indeed, he proved the following

Theorem 1. 1 [7]. There exist a Banach space B and an affine shrinking map S of the closed unit ball U of B into the boundary of U such that S does not have any fixed point.

In view of the above, the following fixed point theorem of D. S. Bridges *et al.* [1] is remarkable.

Theorem 1. 2 [1]. A bounded contractive self-map of R^n has a unique fixed point.

This leads to the following (see [1], *p*. 441) :

Theorem 1. 3 (Brouwer fixed point theorem for contractive maps). A contractive self-map of a compact convex subset of R^n has a unique fixed point.

The purpose of this paper is to extend Theorems 1.2 and 1. 3 for a pair of maps on R^n.

2. Fixed Point Theorems

Two maps $S, T : X \to X$ are said to have a coincidence if there exists a point z in X such that $Sz = Tz$. Let $\mathcal{C}(S, T) = \{z : Sz = Tz\}$. Following Goebel [3] (see also [4], [5]), first we establish an extension of Theorem 1. 2.

Theorem 2.1. Let $S, T : R^n \to R^n$ be such that S is bounded and

$$S(R^n) \subset T(R^n). \tag{1}$$

Suppose that for every $x, y \in R^n$,

$$\left.\begin{array}{c} \| Sx - Sy \| < \| Tx - Ty \| \text{ if } Tx \neq Ty \\ Sx = Sy \text{ if } Tx = Ty. \end{array}\right\} \tag{2}$$

Then S and T have a conicidence. Further, if S and T commute at $z \in C(S, T)$, than S and T have a uniqne common fixed point.

Proof. Define $F : T(R^n) \to T(R^n)$ by $Fa = S(T^{-1}a)$ for each $a \in T(R^n)$ where T^{-1} denotes the inverse image of a under T. To see that F is a properly defined map, observed that for $x \in T^{-1}a$,

$$Fa = \{x : x \in T^{-1}a\} \subset S(R^n), \tag{3}$$

that is, by (1), $Fa \subset T(R^n)$ for every $a \in T(R^n)$. Further, suppose b, $c \in Fa$. Then there exist $x, y \in T^{-1}a$ such that $b = Sx$ and $c = Sy$. Then (2) implies $b = c$. Thus F is a well defined map on $T(R^n)$. Moreover, by (3), F is bounded, since the image of F is contained in $S(R^n)$ and S is bounded.

Now, if $a \neq b$, $a, b \in T(R^n)$, then $T^{-1}a \cap T^{-1}b = \phi$.

So $Tx \neq Ty$ for $x \in T^{-1}a$ and $y \in T^{-1}b$.

Thus

$$\| Fa-Fb \| = \| Sx-Sy \| < \| Tx-Ty \| = \| a-b \| .$$

Hence, by Theorem 1. 2, F has a unique fixed point, *i. e.*, there exists $u \in T(R^n)$ such that $u=Fu$. If $z \in T^{-1}u$ then

$$Sz=S(T^{-1}u) = Fu=u=Tz.$$

If S and T commute at z, then $SSz=STz=TSz$ If $Tz \neq TSz$, then

$$\| Tz-TSz \| = \| Sz-SSz \| < \| Tz-TSz \| ,$$

a contradiction. This proves that $Tz(=Sz)$ is a common fixed point of S and T. The uniqueness of the common fixed point follows from (2).

The following is an extension of Theorem 1. 3.

Theorem 2. 2. Let K be a subset of R^n, and $S, T : K \to K$ such that $T(K)$ is compact and convex. If, for every $x, y \in K$, (2) holds, then S and T have a coincidence. Further, if $STz=TSz$ for every z in $C(S, T)$, then S and T have a unique common fixed point

Proof. An appropriate blend of the proof of Theorem 2. 1 will yield the result

In case $K=R^n$ in Theorem 2 2, we may have the following, which is also a variant of Naimpally *et al* [4, *Cor* 3].

Theorem 2 3. Let (X, d) be a metric space and $S, T : X \to X$ such that $S(X) \subset T(X)$ If $T(X)$ is compact and $d(Sx, Sy) < d(Tx, Ty)$, $Tx \neq Ty$, for $x, y \in X$, then S and T have a coincidence. Further, if $STz=TSz$ for every $z \in C(S, T)$, then S and T have a unique common fixed point.

In [4], the coincidence part of Theorem 2.3 is obtained when S is a multivalued map. One may ask if Theorems 2.1 and 2.2 can be extended to the case when S is a multivalued map. In particular, we have the following :

Question. Let $CL(R^n)$ denotes the set of nonempty closed subsets of R^n, H the (generalized) Hausdorff metric on $CL(R^n)$ induced by the norm of R^n, and $S:R^n \to CL(R^n)$, $T:R^n \to R^n$ such that S is bounded and $S(R^n) \subset T(R^n)$. Suppose that for every $x, y \in R^n$,

$$H(Sx, Ty) < \| Tx - Ty \| \text{ if } Tx \neq Ty$$

$$Sx = Sy \quad \text{if} \quad Tx = Ty.$$

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