Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

SURJECTIVITY OF WEIGHTED Φ -ACCRETIVE OPERATORS

By

H. K. Pathak and Rekha Sharma

Department of Mathematics, Kalyan Mahavidyalaya

Bhilainagar-490006, M. P.

(*Received : October 10, 1992*)

ABSTRACT

In this paper we consider various tvpes of weighted forms of locally ϕ -accretive operators, and obtain surjectivity results including those of Browder [1, 2], Park and Park [5], Park aud Park [6] and Ray [7].

1. INTRODUCTION

Let X be Banach spaces with Y^* the dual of Y, and let $\Phi : X \to Y^*$ be a map satisfying the following:

(1) $\Phi(X)$ is dense in Y^*, and

(2) for such $x \in X$ and each $\alpha \geqslant 0$, $\| \Phi(x) \| \leqslant \| x \|$

and $\Phi(\alpha x) = \alpha\Phi(x)$

A map $P : X \to Y$ is said to be strongly Φ -accretive [1] if there exists a constant $c > 0$ such that, for all $u, v \in X$,

(3) $< Pu - Pv, \Phi(u-v) > \geqslant c \| u - v \|^2$.

The Φ -accretive maps were introduced in an effort to unify the theories for monotone maps (*when* $Y = X^*$) and for accretive maps (when $Y = X$). Surjectivity of those maps was studied by Browder [1, 2], Kirk [4] and Ray [7].

A map $P : X \to Y$ is said to be locally strongly Φ -accretive [4], if, for each $y \in Y$ and $r > 0$, there exists a constant $c > 0$, there exists a constant $c > 0$ such that

(4) if $\| Px - y \| \leqslant r$ then, for all $u \in X$ sufficiently near to x,

$$< Pu - Px,\ \Phi(u - x) > \ \geqslant c \| x - u \|^2.$$

Kirk [4] and Ray [7] extended the surjectivity theorem of Browder [2] to the class of locally strongly Φ -accretive maps by applying the Caristi-Kirk fixed point theorem [3, 4].

For a Banach space Y, we denote by J the duality map from Y into 2^{Y^*} given by

$$J(y) = \{ y^* \in Y^* \mid \| y^* \|^2 = \| y \|^2 = < y, y^* > \}$$

It is well known that, by the Hahn-Banach theorem, $J(Y)$ is not empty for each $y \in Y$, J is single valued whenever Y^* is strictly convex, and J is uniformly continuous on bounded subsets of Y whenever Y^* is uniformly convex. Note that for any $y^* \in J(y)$ we have

(5) $\| y \|^2 \leqslant \| z \|^2 = 2 < z - y, y^* >$ for any $z \in Y$

Lemma. [5]. For any $y \in Y$, $y \neq 0$, $y^* \in J(y)$, and $\epsilon > 0$ there exists an $h \in X$ such that $\| h \| \geqslant 1$ and

$$\| \Phi(h) - y^* \| y \|^{-1} \| < \epsilon$$

The duality map J is said to be lower semicontinuous if the following condition holds:

(6) If $\lim_{n \to \infty} y_n = y$ and $y^* \in J(y)$, then there exists a sequence $\{ y_n^* \}$ such that

$$y_n^* \in J(y_n) \text{ and } \lim_{n \to \infty} y_n^* = y^*$$

Theorem. Let X and Y be Banach spaces and $P: X \to Y$ a locally Lipschitzian and locally strongly Φ -accretive map. If the duality map J of Y is lower semicontinuous and $P(X)$ is closed, then $P(X) = Y$.

Motivated by the work [6] of Park and Park, in the present paper the constant C in definitions of (locally) stronglg Φ -accretive maps is replaced by a certain functional value, and we show that some known surjectivity theorems on (locally) strongly Φ -accretive maps can be extended to this general setting.

2. Results. Let $C : [0, \infty) \to (0, \infty)$ be a continuous nonincreasing function.

Let us consider the following types of weighted (locally) Φ -accretive maps.

(i) For any $u, v \in X$,

$$< Pu-Pv, \Phi(u-v) \geqslant c(\| u-v \| \; \| h\|^{-1}) \| u-v\|^2$$

(ii) For any $x \in X$, there exists an $\epsilon > 0$ such that for any $u \in \bar{B}\ (x;\epsilon)$

$$< Pu-Px, \Phi(u-x) \geqslant c(\| u-x \| \; \| h\|^{-1}) \| n-x\|^2$$

(iii) For any $x \in X$, there exists an $\epsilon > 0$ such that for any $u, v \in \bar{B}\ (x; \epsilon)$

$$< Pu-Pv, \Phi(u-v)> \geqslant c(max \| u \| , \| v \| \; \| h\|^{-1}) \| u-v\|^2$$

(iv) For any $x \in X$, there exists and $\epsilon > 0$ such that for any $u \in \hat{A}\ (x; \epsilon)$,

$$< Pu-Px, \Phi(u-x)> \geqslant c(\| x \| \; \| h\|^{-1}) \| u-x\|^2.$$

Moreover, we state the following type of weighted (locally) Φ -accretive maps.

(v) For each $y \in Y$ and $r > 0$ there exis's a non-increasing function $c : [0, \infty) \to (0, \infty)$ such that if $\| Px-y \| \leqslant r$ then for all $u, v \in X$ sufficiently near to x.

$$< Pu-Pv, \Phi(u-v) > \geqslant c(\| u-v \| \; \| h\|^{-1}) \| u-v\|^2$$

(vi) For each $y \in Y$ and $r > 0$ there exists a non-increasing function

$c : [0, \infty) \to (0, \infty)$ such that if $\| Px - y \| \leqslant r$ then for all $u \in X$ sufficiently near to x.

$$< Pu - Px, \Phi(u-x) > \geqslant c(\|x\| \; \|h\|^{-1}) \|u-x\|^2$$

Note that (i) $\Rightarrow$ (ii) $\Rightarrow$ (v) $\Rightarrow$ (vi) and (iii) $\Rightarrow$ (v) $\Rightarrow$ (vii).

Further more, if P is locally strongly Φ -accretive, then P satisfies (vi). Now we have our main result :

Theorem 1. Let X and Y be Banach spaces and $P : X \to Y$ be a locally Lipschitzian map satisfying condition (vi). If the duality map J of X is l. s. c. and $P(x)$ is closed, then $P(x) = Y$

Proof. It suffices to prove that $P(x)$ is open. Since $J(y) \neq \Phi$ for each $y \in Y$, we can choose $y^* \in J(y)$.Since P is locally Lipschitzian for any given $x_0 \in X$, we can choose ϵ_1 such that P is Lipschitzian with a constant M on $\bar{B}(x_0, 2\epsilon_1)$. By (vi), we choose a non-increasing function c and an $\epsilon_2 > 0$ such that, for all

$Px \in \bar{B}(Px_0, 2M\epsilon_1)$ and $u \in \bar{B}(x_0, 2\epsilon_2)$,

$$< Px - Pu, \Phi(x-u) > \geqslant c(\|x\| \; \|h\|^{-1}) \|x-u\|^2$$

Let $r = \min(c \|x_0\| \; \|h\|^{-1} . \epsilon/2, M\epsilon)$, where $\epsilon = \min(\epsilon_1, \epsilon_2)$. Let $a = c(\|x_0\| \; \|h\|^{-1}) > 0$. Suppose there exists $y \in \bar{B}(P\,0, r)$ and $y \notin P(x)$. Then we get a contradiction by applying the Caristi-Kirk fixed point theorem [3].

Now we let $d = dist(y, P(x)) > 0$ and $D = \{x \in \bar{B}(x_0, \epsilon) \mid \|y - Px\| \leqslant r\}$. Since $x_0 \in D, D$ is a nonempty complete metric space. For any given $x \in D$,by the lemma,we can choose a $h \in X, \|h\| \geqslant 1$, such that

$$\| \Phi(h) - (y-Px)^* \|y-Px\|^{-1} \leqslant a/2M.$$

Since $\|Px - Px_0\| \leqslant \|Px - y\| + \|y - Px_0\| \leqslant 2r \leqslant 2M\epsilon_1$

We have

$$< Px_t - Px, \Phi(x_t - x) > \geqslant c(\| x \| \, \| h \|^{-1}) \| x_t - x \|^2$$

for sufficiently small $t > 0$, where $x_t = x + th$.

Since c is non-increasing,

$$< Px_t - Px, \Phi(x_t - x) > \geqslant c(\| x_0 \| \, \| h \|^{-1}) \| x_t - x \|^2,$$

which implies

$$< Px_t - Px, \Phi(h) > \geqslant at \| h \|^2.$$

Since P is Lipschitzian with constant M on $B(x_0; 2\epsilon_1)$ for small $t > 0$, we have

$$< Px_t - Px, \Phi(h) > \geqslant \frac{a}{M} \| Px_t - Px \| .$$

Now, again applying the lemma,

$$< Px_t - Px, (y - Px)^* >$$

$$= < Px_t - Px, \| y - Px \| \Phi(h) - \| y - Px \| \Phi(h) + (y - Px)^* >$$

$$\geqslant < Px_t - Px, \Phi(h) > \| y - Px \| - < Px_t - Px, \Phi(h) - (y - Px)^* (y - Px)^{-1}) > \| y - Px \|$$

$$\geqslant \frac{a}{m} \| y - Px \| \, \| Px_t - Px \| - \| y - Px \| \, \| Px_t - Px \| \, \| \Phi(h) - (y - Px)^* (y - Px)^{-1} \|$$

$$\geqslant \frac{a}{M} \| y - Px \| Px_t - Px \| - \frac{a}{2M} \| y - Px \| \, \| Px_t - Px \| .$$

With this inequality and (5), we estimate $\| y - Px_t \|$

$$\| y - Px_t \|^2 \leqslant \| y - Px \|^2 - 2 < Px_t - Px, (y - Px_t)^* >$$

$$= \| y - Px \|^2 - 2 < Px_t - Px, (y - Px)^* - (y - Px)^* + (y - Px_t)^* >$$

$$\leqslant \| y - Px \|^2 - \frac{ad}{M} \| Px_t - Px \| + 2 \| Px_t - Px \| (y - Px)^* - (y - Px_t)^* \|$$

Since the duality map J is *l. s. c.* we can select $t > 0$ such that

$$\| (y - Px)^* - (y - Px_t)^* \| \leqslant ad/4M.$$

Therefore,

$$\| y - Px_t \|^2 \leqslant \| y - Px \|^2 - \frac{ad}{2M} \| Px_t - Px \| .$$

Hence $Px_t \in \bar{B}(y; r)$ and $x_t \in B(x_0; 2\epsilon_1)$. So,

$$\| x_t - x_0 \| \leqslant \frac{1}{c(\| x_0 \| \; \| h \|^{-1})} \| Px_t - Px_0 \|$$
$$\leqslant \frac{1}{c(\| x_0 \|)} (\| Px_t - y \| + \| y - Px_0 \|)$$
$$\leqslant \frac{2r}{c(\| x_0 \|)}$$

i. e. $x_t \in \bar{B}(x_0, \epsilon)$ and $x_t \in D$. Furthermore by (7), $\| Px_t - Px \| \geqslant a \| x_t - x \|$. If we let $g : D \to D$ such that $g(x) = x_t \in D$ and $\Psi(x) = (2m/a^2 d) \| y - Px \|^2$ then we have $\| gx - x \| \leqslant \Psi(x) - \Psi(g(x))$ for $x \in D$. Since Ψ is the continuous map from the complete metric space D into the non-negative reals. Hence, by the Caristi-Kirk fixed point theorem, g has a fixed point in D. Since $\| x_t - x \| = t \| h \| \neq 0$ this is a contradiction. When $\| h \| = 1$, we obtain the following

Corollary: Let X and Y be Banach space $P : X \to Y$ be a locally Lipschitzian map satisfying conditio

(vi)′ $\langle Pu - Px, \Phi(u - x) \rangle \geqslant c(\| x \|) \| u - x \|^2$.

If the duality map J of X is *l. s. c.* and $P(X)$ is closed, then $P(X) = Y$.

Remark. If we replace (vi) with (v) in Theorem 1, then we may obtain the same conclusion. Note also that Theorem 1 extends results of ([1], [2], [5], [6], [7]).

Acknowledgements

The authors are thankful to the U.G.C., New Delhi for providing the financial support to the project.

REFERENCES

[1] F. E Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, *Proc. Sympos. Pure Math.*, **18** Part (2), Amer. Math. Soc., Providence, R. I., 1976.

[2] F. E. Browder, Normal solvability and ϕ accretive mappings of Banach spaces, *Bull. Amer. Math. Soc.* **78** (1972), 186–192.

[3] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, *Trans. Amer. Mnth. Soc.* **215** (1976), 241-251.

[4] W. A. Kirk, Caristi's fixed point theorem and the theory of normal solvability. *Proceedings, Conference on Fixed Point Theory and its Applications* (Dalhousie Univ , June 1975), Academic Press, New York (1976), 109-120.

[5] J. A. Park and Sehie Park, Surjectivity of ϕ -accretive operators, *Proc. Amer. Math Soc.* **90** (1984), 289-292.

[6] Sehie Park and J. A. Park, Surjectivity of Generalized ϕ -Accretive Operators, *Proc. Sympos. Pure Math.*, **45** (1986), Part 2.

[7] W. O. Ray, Phi accretive operators and Ekeland's theorem, *J. Math. Anal Appl.* **88** (1982), 566-571.